Ring Extensions Involving Amalgamated Duplications

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By

Timothy S. Long
Master of Science
George Mason University, 2012
Bachelor of Science
George Mason University, 2005

Director: Dr. Jay Shapiro, Associate Chair
Department of Mathematical Sciences

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George Mason University
Fairfax, VA

Dedication

I dedicate this dissertation to my mother, the embodiment of unconditional love and support.

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Abstract

RING EXTENSIONS INVOLVING AMALGAMATED DUPLICATIONS

Timothy S. Long, PhD

George Mason University, 2014

Dissertation Director: Dr. Jay Shapiro

We investigate properties of the amalgamated duplication of a ring along an ideal (which we will usually refer to simply as a *bowtie ring*), recently studied most notably by D'Anna, Fontana, and Finocchiaro. If I is an ideal of a ring R, then the basic construction of the amalgamated duplication of R with I is $\{(r, r+i) | r \in R, i \in I\}$. For rings R, R' and an ideal I of R' the general construction involves a ring homomorphism $f : R \to R'$ and is given by $\{(r, f(r)+i) | r \in R, i \in I\}$. We will consider in particular different types of ring extensions in which the base ring and the extension ring are both amalgamated duplications.

In Chapter 2 we provide some introductory lemmas to be employed later. After describing when a construction will be a ring and when two such constructions will be equal, we see how set operations affect bowtie rings, observing what rings result from taking sums, composites, and intersections of bowtie rings. We use these results conversely to examine when an arbitrary ring can be written as a sum, composite, or intersection of certain types of bowtie rings.

The subsequent chapter looks at the partially ordered set of intermediate rings between two bowtie rings (where the partial order is inclusion as subrings). For any given ring R we find an order-isomorphism between the subrings of $R \times R$ containing R (embedded into

$R \times R$ along the diagonal) and the ideals of R; in particular, if R is an arithmetical ring, this order-isomorphism is a lattice isomorphism. We show for any ideals $I \subset J$ of R, that the correspondence holds for the rings lying between $R \bowtie I$ and $R \bowtie J$ and the ideals of R lying between I and J. For a ring extension $R \subset T$ where J is an $f(T)-$subalgebra of T, a similar correspondence is found between the intermediate rings of $R \bowtie^f J \subset T \bowtie^f J$ and the intermediate rings of $R \subset T$. In this chapter we also consider minimal ring extensions (i.e., ring extensions $R \subset T$ with no rings lying strictly between the two), and we determine when an extension involving two bowtie rings is minimal.

Next, in the Flat Epimorphisms chapter (Chapter 4) we characterize when a homomorphism of one bowtie ring into another bowtie ring is a flat epimorphism (thus a ring extension of the form usually studied in this document is a special case, where the map is simply the inclusion map). Specifically, if $f : R \to T$ a ring homomorphism, and I, J are an $f(R)-$subalgebra and an $f(T)-$subalgebra of T, respectively, with $I \subseteq J$, then defining $f' : R \bowtie^f I \to T \bowtie^f J$ as $f'((t, t + j)) = (f(t), f(t) + f(j))$ we find that f' is a flat epimorphism if and only if f is a flat epimorphism and $J = f(I)T$. As an application we show that this result can be used to construct flat epimorphisms that are not minimal extensions.

In the Complemented Rings and Related Topics chapter, we observe situations where a bowtie ring will have some related properties known as Property A, the (a.c.), $Min(R)$ being compact, and the ring R being complemented.

It is known that if I is a regular ideal of a ring R, then $tq(R \bowtie I) = tq(R) \times tq(R)$ (where $tq(S)$ is the total quotient ring of a given ring S). After analyzing the above properties in the case where I is regular, we supplement this result by observing the structure of $tq(R \bowtie I)$. With only the information provided in Chapter 3 we see that $tq(R \bowtie I) = tq(R) \bowtie J$ for some (possibly improper) ideal J of $tq(R)$, and this is sufficient information to show that $R \bowtie I$ is complemented if and only if R is complemented. We complete the chapter by describing the exact form of $tq(R \bowtie I)$ for any ideal I of any ring R.

Finally, in Chapter 6 we consider integrality of extensions of bowtie rings. We see that

$R \bowtie I \subset R \bowtie J$ is always an integral extension (for any ideals $I \subset J$ of a ring R) and find cases of bowtie ring extensions that are not integral. Further, we use the description of the total quotient ring of a bowtie ring (found in Chapter 5) to exactly describe the integral closure of a bowtie ring. Later we show that if R is an integrally closed subring of a ring T and J is a common ideal to both of these rings, then $R \subset T$ is a normal pair if and only if $R \bowtie J \subset T \bowtie J$ is a normal pair.

We study various extension ring properties closely related to integrality, including lying-over, going-up, incomparability, and going-down. The first three of these properties give predictable results. However, given ideals $I \subseteq J$ of rings $R \subseteq T$, respectively, we find that the extension $R \bowtie I \subset T \bowtie J$ satisfying going-down is equivalent to the properties that $R \subseteq T$ satisfies going-down and for every pair of primes $P \subset Q$ of T, $I \nsubseteq P$ and $J \nsubseteq Q$ implies that $I \nsubseteq Q$.

Chapter 1: Introduction

1.1 Notation

Throughout the following document we let every ring be commutative with identity element $1 \neq 0$. We assume that all ring homomorphisms (and thus all ring extensions) are unital (i.e., for any ring homomorphism $f : R \to S$ we have that $f(1_R) = 1_S$ and for any ring extension $R \subseteq T$ we have $1_R = 1_T$). Let R be a ring. We let $Spec(R)$, $Max(R)$, and $Min(R)$ denote the set of prime ideals, maximal ideals, and minimal prime ideals of R, respectively. We use $dim(R)$ to denote the (Krull) dimension of R.

An element x of R is *nilpotent* if $x^n = 0$ for some nonnegative integer n. We will use $Nil(R)$ to denote the *nilradical* of R, i.e., the set of all nilpotent elements of R (which forms an ideal in R); we say R is *reduced* if $Nil(R) = 0$. We use $Z(R)$ to denote the set of zero-divisors in R, $Reg(R)$ to denote the *regular elements* of R (i.e., the non-zero-divisors in R), and we use $tq(R)$ to denote $R_{Reg(R)} := \left\{\frac{r}{s} | r \in R, s \in Reg(R)\right\}$, the *total quotient ring* of R.

If T is an extension ring of R we say an element $t \in T$ is *integral over* R if it satisfies some polynomial with coefficients in R and leading coefficient 1. We use $\overline{R}^T$, to denote the *integral closure of* R *in* T (i.e., the set of elements in T that are integral over R). This set forms a subring of T; in particular, the sum of two integral elements is integral. We remove the superscript and write $\overline{R}$ to denote the *integral closure of* R (that is, the integral closure of R in its total quotient ring). We say R is *integrally closed in* T (resp., *integrally closed*) if $\overline{R}^T = R$ (resp., $\overline{R} = R$).

Given an ideal $I \subseteq R$ we will write $Ann_R(I)$ for the *annihilator* of I, i.e., $Ann_R(I) := \{r \in R | rI = 0\}$. If $I = (x)$ is principal we will write $Ann_R(x)$ for the annihilator of (x)

(and also refer to this as the *annihilator of* x). An ideal I is called *regular* if it contains a regular element, and *dense* if $Ann_R(I) = 0$. Clearly every regular ideal is dense. We say R is *quasilocal* if it has a unique maximal ideal m; in this case we often write R as (R, m).

Given rings R, R' and a ring homomorphism $f : R \to R'$ we will use the notation $R^{\Delta} := \{(r, r)|r \in R\}$ as a subring of $R \times R$ (resp., $\Gamma(f) := \{(r, f(r)|r \in R\}$ as a subring of $R \times R'$) and describe this as the *diagonal image of* R in $R \times R$ (resp., in $R \times R'$). This is a subring of $R \times R$ (resp., $R \times R'$) isomorphic to R.

We let $\mathbb{N}$ denote the positive integers and identify these as the "natural numbers." Finally, we use $\subset$ to denote strict containment.

1.2 Nonstandard Conventions

Remark 1.2.1. Let R be a ring. For the remainder of this dissertation we will adopt the following unconventional definition for an *R-algebra*: Let A be an R-module where A is also a multiplicative semigroup, possibly without 1 (we retain all the usual axioms for A to be an R-algebra except that we allow the possibility A *might not* have a multiplicative identity.) Under this same definition, if B is an $R-$algebra contained in A we will say that B is an *R-subalgebra* of A.

Remark 1.2.2. In this dissertation when we simply refer to an *ideal* I of a ring R, we allow that possibilities the $I = 0$ or $I = R$ (unless otherwise noted). Similar allowances apply to "subalgebras" (as they are defined in the previous remark).

1.3 Background

In 1932 Dorroh gave an algebraic construction that would allow any ring to be embedded into a ring with identity [D3]. Specifically, for a ring R without identity, the set $\{(n, r)|n \in \mathbb{Z}, r \in R\}$ with multiplication defined by $(n, r)(m, s) = (nm, ns + rm + rs)$ is a ring with identity $(1, 0)$ into which R embeds via the map $r \mapsto (0, r)$. The non-unital ring

$\{(0,r)|r \in R\} \cong R$ in fact becomes an ideal in this new ring. The new unital ring construction is sometimes referred to as a *Dorroh ring*, and we will adhere to this terminology.

In [N] (also see [N2]) M. Nagata- the mathematician best known for solving Hilbert's 14th problem- considered a specific case of Dorroh's construction, introducing the *idealization* of a module. Idealization has become an invaluable tool in algebra for constructing examples and counterexamples of numerous types of rings. It is defined as follows: for a commutative ring R (with identity 1) and an $R-$module M, we define the *idealization of M (over R)*, or the *trivial extension of R by M*, to be the set $R \ltimes M := \{(r,m)|r \in R, m \in M\}$ (also occasionally written as $R(+)M$) with addition defined componentwise as in the $R-$module $R \oplus M$, but with multiplication given by $(r,m)(s,n) = (rs, rn + sm)$. This produces a ring with identity element $(1,0)$. Here R embeds into $R \ltimes M$ via the map $r \mapsto (r,0)$, and M embeds into it via the map $m \mapsto (0,m)$; the image of M is an ideal of $R \ltimes M$ whose square is zero.

In this paper we discuss a very similar construction, called an *amalgamated duplication of a ring along an ideal* or simply a *bowtie ring*, as well as a standard generalization of this construction. The original construction involves an ideal I of a ring R, and is given by the construction $R \bowtie I := \{(r, r+i)|r \in R, i \in I\}$ where addition and multiplication are componentwise as in $R \times R$. This can be considered as an extension ring of R via the embedding of R into $R \bowtie I$ given by $r \mapsto (r,r)$. Another way to view the ring $R \bowtie I$ is by taking the set $\{(r,i)|r \in R, i \in I\}$ with addition defined pointwise as in the $R-$module $R \oplus I$, and with multiplication given by $(r,i)(s,j) = (rs, rj + si + ij)$. The original construction is isomorphic to this one via the map $(r, r+i) \mapsto (r,i)$ (and it follows that R embeds into this ring via the map $r \mapsto (r,0)$). In the second form it is easy to see that if $I^2 = 0$, then the bowtie ring is isomorphic to Nagata's idealization of the $R-$module I. Alternatively, letting I be an ideal of $R = \mathbb{Z}$ and taking I to be our ring without identity in Dorroh's construction, we see easily that $R \bowtie I$ is also isomorphic to Dorroh's ring with identity via the map $(r, r+i) \mapsto (r,i)$ (respectively, via the map $(r,i) \mapsto (r,i)$, in the second form of $R \bowtie I$).

Following [DFF2] (with only slightly modified notation), the more general case of the bowtie construction is described here. Let R, R' be rings with I an ideal of R' and let $f : R \mapsto R'$ be a ring homomorphism. Then we define $R \bowtie^f I = \{(r, f(r) + i) | r \in R, i \in I\}$. This construction is referred to as the *amalgamated duplication of the ring R along the ideal I with respect to f*. It is a clear generalization of the amalgamated duplication of a ring along an ideal as described above. Further, this construction simultaneously generalizes some important constructions in commutative ring theory, including Dorroh's embedding into a ring wih identity and Nagata's idealization, the classical $D + M$ construction, and the $A + XB[X]$ and $A + XB[[X]]$ constructions (cf. [DFF2]).

Although similar constructions to these "bowtie rings" were briefly employed by Shores in [S2] and Corner in [C], it was D'Anna and Fontana who gave the structure much analysis more recently in [D], [DF], and [DF2] (with a small error in [D] corrected by Shapiro in [S1]). D'Anna and Fontana were joined by Finocchiaro to study the construction further in [DFF] and [DFF2], and Finocchiaro included many of these results along with new ones in his PhD dissertation [F]. The construction- in both the general and non-general cases- has gained much interest in the field of commutative algebra over the last decade. We wish to expand on the literature by investigating how these ring constructions behave in the context of ring extensions.

1.4 Amalgamated Duplications ("Bowtie Rings")

For a ring of the form $R \bowtie I$ or $R \bowtie^f I$ as described in the previous section, multiple names have been used, including an amalgamated duplication of a ring along an ideal, an amalgamated duplication ring, and- mainly in a noncommutative context- a Dorroh ring or an ideal extension. For brevity, we will adhere to the terminology *bowtie ring*, as used in [DS4]. For reference we recall the definitions in the following remark.

Remark 1.4.1. Let I be an ideal of a ring R, and define $R \bowtie I := \{(r, r + i) | r \in R, i \in I\}$. Henceforth, any ring of the form $R \bowtie I$ (I and ideal of the ring R, or more generally an

$R-$algebra) will be referred to as a *simple bowtie ring*, or a *simple bowtie extension of* R. In our most basic context we will investigate ring extensions of the form $R \bowtie I \subset T \bowtie J$ where $R \subseteq T$ is a ring extension and I, J are ideals of R, T, respectively, such that $I \subseteq J$.

We will define the more general construction with algebras rather than ideals; the reasons for this will become evident in Chapter 3. If $f : R \to R'$ is a ring homomorphism and I is an $f(R)-$subalgebra of R', then a ring of the form $R \bowtie^f I := \{(r, f(r) + I) | r \in R, i \in I\}$ will be referred to as a *general bowtie ring*, or a *general bowtie extension of* R. Given a ring homomorphism $f : T \to T'$ and a subring R of T we will slightly abuse notation by also simply writing f for the map f restricted to R. Our primary focus will be investigating ring extensions of the form $R \bowtie^f I \subset T \bowtie^f J$, where $f : T \to T'$ is a ring homomorphism, $R \subset T$ is a ring extension, I is an $f(R)-$subalgebra of T', and J is an $f(T)-$subalgebra of T' such that $I \subseteq J$. Note that this generalizes ring extensions of the form described in the previous paragraph (which belong to the special case where $T = T'$ and f is simply the identity map).

Remark 1.4.2. We allow R to embed into $R \bowtie I$ (resp., $R \bowtie^f I$) along the *diagonal* $R^\Delta := \{(r, r) | r \in R\} \cong R$ (resp., $\Gamma(f) := \{(r, f(r)) | r \in R\} \cong R$). Thus we have that $R^\Delta \subseteq R \bowtie I \subseteq R \times R$ (resp., $\Gamma(f) \subseteq R \bowtie^f I \subseteq R \times R'$). Note that the diagonal image and cross product can be viewed as bowtie rings as well, for $R^\Delta = R \bowtie 0$ and $R \times R = R \bowtie R$ (and generally, $\Gamma(f) = R \bowtie^f 0$ and $R \times R' = R \bowtie^f R'$).

In the study of bowtie rings of the form $R \bowtie^f I$, the two ring extensions that ostensibly garner the most attention are the extensions $\Gamma(f) \subseteq R \bowtie^f I$ and $R \bowtie^f I \subseteq R \times R'$. We note that the general extensions $R \bowtie^f I \subset T \bowtie^f J$ studied in this paper generalize both of these extensions by the above remark. Namely, setting $I = 0$, $T = R$, our extension has the form $\Gamma(f) \subseteq R \bowtie J$. Alternatively, setting $T = R, J = R'$, our extension becomes $R \bowtie^f I \subset R \bowtie R' = R \times R'$. This paper also generalizes any extensions of rings that can be described by the general bowtie construction, for instance the extension $A + XB[X] \subset A + XB[[X]]$ (where $A \subseteq B$ is any ring extension).

Example 1.4.3. For perhaps the easiest example of an extension relative to this paper let $R = \mathbb{Z}, I = 4\mathbb{Z}, J = 2\mathbb{Z}$. Then $R \bowtie I$ is the set of all pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that x and y differ by a multiple of 4, and $R \bowtie J$ is the set of pairs that differ by an even number. Clearly $R \bowtie I \subset R \bowtie J$, so we have an extension of simple bowtie rings.

Example 1.4.4. Let R be a ring and set $R' = R[[X]]$. If we set $I = XR[X]$ and $J = XR[[X]]$, and f as the natural embedding from R into R', we have an extension of general bowtie rings where the ring R stays fixed, $R \bowtie^f I \subset R \bowtie^f J$. In this sense the extension of bowtie rings is simply another way to write the extension $R[X] \subset R[[X]]$ (cf. [DF2, Example 2.5]).

As we will see, extensions where the ring R or $f(R)-$subalgebra I stays fixed will be critical in our study of minimal ring extensions. When the subalgebra stays fixed we will denote it by J rather than I, noting the importance of the larger subalgebra in the general extension $R \bowtie^f I \subset T \bowtie^f J$: specifically, J will always be an $f(R)-$algebra, but there is no guarantee that I will be an $f(T)-$algebra.

Example 1.4.5. Let $R \subset T$ be a ring extension. Consider the rings $R' = R + XT[X]$ and $T' = T[X]$. Then setting $J = XT[X]$ and $f : T \to T[X]$ as the natural embedding, we have an example of an extension of general bowtie rings where the ideal J stays fixed, $R \bowtie^f J \subset R' \bowtie^f J$.

Example 1.4.6. Finally, for a straightforward example where neither the ring R, nor the $f(R)-$subalgebra I stays fixed, let $R = \mathbb{Z}, T = \mathbb{Q}, T' = \mathbb{Q}[X], I = X\mathbb{Z}[X], J = X\mathbb{Q}[X]$. Now set $f : T \to T'$ as the natural embedding. Then the extension $R \bowtie^f I \subset T \bowtie^f J$ can be viewed another way to write the extension $\mathbb{Z}[X] \subset \mathbb{Q}[X]$.

1.5 Some Known Results

Many properties of bowtie rings have been analyzed over the last decade and the majority of these were investigated by D'Anna, Fontana, and Finocchiaro (cf. [DF], [DFF], [DFF2]).

In this section we record a few important known results, in particular those that will be relevant to our study of bowtie rings.

The following proposition is essentially condensed version of [DFF2, Proposition 5.1].

Proposition 1.5.1. *Let $f : R \to R'$ be a ring homomorphism, I an ideal of R', and define $R \bowtie^f I = \{(r, f(r) + i | r \in I, i \in I\}$.*

1. *$R \bowtie^f I$ contains (an isomorphic image of) R as a subring, namely $\Gamma(f) :=$ $\{(r, f(r)) | r \in R\}$ into which R maps via the isomorphism $r \to (r, f(r))$.*
2. *For any ideal J of R, the set $J \bowtie^f I = \{(j, f(j) + i | j \in J, i \in I\}$ is an ideal of $R \bowtie^f I$. Further, we have the following canonical isomorphism:*

$$\frac{R \bowtie^f I}{J \bowtie^f I} \cong \frac{R}{J}.$$

3. *The sets $\{0\} \times I$ and $f^{-1}(I) \times \{0\}$ are ideals of $R \bowtie^f I$ and the following canonical isomorphisms hold:*

$$\frac{R \bowtie^f I}{\{0\} \times I} \cong R, \qquad \frac{R \bowtie^f I}{f^{-1}(I) \times \{0\}} \cong f(R) + I.$$

4. *The sets $f^{-1}(I) \times I$ and I are ideals of the rings $R \bowtie^f I$ and $f(R) + I$, respectively, and the following isomorphism holds:*

$$\frac{R \bowtie^f I}{f^{-1}(I) \times I} \cong \frac{f(R) + I}{I}.$$

In particular, if f is surjective then this gives the following isomorphism

$$\frac{R \bowtie^f I}{f^{-1}(I) \times I} \cong \frac{R'}{I}.$$

It is not hard to see that when $I \neq 0$, then the simple bowtie ring $R \bowtie I$ is never an integral domain (since for any nonzero $i \in I$, we have $(i,0)(0,i) = (0,0)$). However it is possible for a general bowtie ring $R \bowtie^f I$ to be a domain even if $I \neq 0$ and R is not a domain.

Proposition 1.5.2. *[DFF2, Proposition 5.2] With the notation of Proposition 1.5.1, if $I \neq 0$, then the following are equivalent.*

1. *$R \bowtie^f I$ is an integral domain.*

2. *$f(R) + I$ is an integral domain and $f^{-1}(I) = \{0\}$.*

In particular, if R' is an integral domain and $f^{-1}(I) = \{0\}$, then $R \bowtie^f I$ is an integral domain.

Given an ideal I of a ring R, it is clear that $R \bowtie I$ is reduced if and only if R is reduced. This contrasts Nagata's $R \ltimes I$ construction, which will never be reduced (assuming $I \neq 0$). It is also possible that the general bowtie ring $R \bowtie^f I$ will not be reduced, even if R is.

Proposition 1.5.3. *[DFF2, Proposition 5.4] With the notation of Proposition 1.5.1, the following conditions are equivalent.*

1. *$R \bowtie^f I$ is a reduced ring.*

2. *R is a reduced ring and $Nil(R') \cap I = \{0\}$.*

If R and R' are reduced, then $R \bowtie^f I$ is reduced. If $R \bowtie^f I$ is reduced and I is a radical ideal, then R and R' are reduced.

Numerous ring properties are equivalently satisfied by $R \bowtie I$ and R, including R being reduced, R being quasilocal, and R being Noetherian. In the context of general bowtie rings, these conditions may become slightly more complicated.

Proposition 1.5.4. *[DFF2, Proposition 5.6] With the notation of Proposition 1.5.1, the following conditions are equivalent.*

1. *$R \bowtie^f I$ is a Noetherian ring.*

2. *R and $f(R)+I$ are Noetherian rings.*

Concerning chains of ideals, we also know the following result on Krull dimension.

Proposition 1.5.5. *[DFF, Proposition 4.1] With the notation of 1.5.1, then $dim(R \bowtie^f I) = max\{dim(R), dim(f(R)+I)\}$. In particular, if f is surjective, then $dim(R \bowtie^f I) = max\{dim(R), dim(R')\} = dim(R)$.*

Corollary 1.5.6. *If I is an ideal of a ring R, then $dim(R \bowtie I) = dim(R)$.*

The following combines [DFF, Proposition 3.1] and [DFF, Proposition 3.4].

Proposition 1.5.7. *Let $f : R \to R'$ be a homomorphism of rings, and let $f^{-1}(I)$, I be regular ideals of R, R', respectively. Then*

1. *$tq(R \bowtie^f I) \cong tq(R) \times tq(R')$, and*

2. *$\overline{R \bowtie^f I} = \overline{R} \times \overline{f(R)+I}$.*

The next lemma follows from [DF2, Proposition 2.2]. Note that the cases for $I = 0$ and $I = R$ hold trivially.

Lemma 1.5.8. *Let I be an ideal of a ring R. Let P be a prime ideal of R and set*

- $P_0 = \{(p, p+i) | p \in P, i \in I \cap P\}$,
- $P_1 = \{(p, p+i) | p \in P, i \in I\}$ *("form 1"), and*
- $P_2 = \{(p+i, p) | p \in P, i \in I\}$ *("form 2").*

(a.) If $I \subseteq P$, then $P_0 = P_1 = P_2$ is a prime ideal of $R \bowtie I$ and it is the unique prime ideal of $R \bowtie I$ lying over P.

(b.) If $I \nsubseteq P$, then $P_1 \neq P_2$, $P_1 \cap P_2 = P_0$, and P_1 and P_2 are the only prime ideals of $R \bowtie I$ lying over P.

(c.) The extension $P(R \bowtie I)$ of P in $R \bowtie I$ coincides with $\{(p, p+i) | p \in P, i \in IP\}$ and, moreover, $\sqrt{P(R \bowtie I)} = P_0$.

Furthermore, every prime of $R \bowtie I$ can be written in form 1 or form 2 for some prime $P \in Spec(R)$.

Proof. The majority of this lemma is simply [DF2, Proposition 2.2]. The final statement then follows from the rest of the lemma and [D, Remark 1(a)] because the integral extension $R \bowtie I$ of $R (\cong R^{\Delta})$ satisfies lying-over [K, Theorem 44].

□

Corollary 1.5.9. *Let I be a proper ideal of a ring R. Then R is quasilocal (with unique maximal ideal M) if and only if $R \bowtie I$ is quasilocal (with unique maximal ideal $\{(m, m+i) | m \in M, i \in I\}$).*

The corollary above is one of the only results in this document that requires that I be proper. Note that if $I = R$ and R is quasilocal with unique maximal ideal M, then $R \bowtie I = R \bowtie R = R \times R$ is no longer quasilocal, now containing the two distinct maximal ideals $M \times R$ and $R \times M$.

Finally, we make a quick note on localization. This proposition was presented in [D, Proposition 2.7]. An alternate proof for the second statement was given in [DS3, Proposition 4.4(b.)].

Proposition 1.5.10. *[D, Proposition 2.7] Let R, I, P, P_1, P_2 be as in Lemma 1.5.8. If $I \subseteq P$, then $(R \bowtie I)_{P_i} \cong R_P \bowtie I_P$ for $i = 1, 2$. If $I \not\subseteq P$, then $(R \bowtie I)_{P_i} \cong R_P$ for $i = 1, 2$.*

We will expand on Proposition 1.5.10 later, when we will need a more general version in the study of flat epimorphisms (see Proposition 4.2.9).

Chapter 2: Preliminary Results

2.1 Basic Lemmas

We will use the present chapter as storage for many lemmas that will be used in this manuscript, in particular those that will be employed multiple times. No theorems will be presented in this chapter, but each of the lemmas will be utilized at some further point in the document.

Before examining constructions of bowtie rings from existing rings in the next section, we present the following lemma to ensure that these constructions will indeed be rings themselves, and the subsequent lemma to ensure that each such construction is unique.

Lemma 2.1.1. *Let $f : R \to R'$ be a ring homomorphism, and let K be a subset of R' with the same operations of addition and multiplication defined. Then $R \bowtie^f K := \{(r, f(r) + k) | r \in R, k \in K\}$ is a ring if and only if K is an $f(R)-$subalgebra of R'.*

Proof. If K is an $f(R)-$subalgebra of the ring R', then the set described is simply the bowtie extension $R \bowtie^f K$ of R.

Conversely, define $R \bowtie^f K := \{(r, f(r) + k) | r \in R, k \in K\}$, and assume that this set forms a ring. By assumption K is contained in R'. We know that $R \bowtie^f K$ as defined here contains the set $\{(0, k) | k \in K\}$, and basic calculations (noting the definition of the ring $R \bowtie^f K$) show that this set is closed under addition and multiplication so that K itself is closed under these operations. Since the ring $R \bowtie^f K$ contains $(0, 0)$, it follows that K contains 0, and for any element $k \in K$, the element $(0, k)$ must have an additive inverse in $R \bowtie^f K$, say $(s, f(s)+j)$, so that clearly $f(s) = s = 0$ and then $j = -k$ lies in K by definition of $R \bowtie^f K$. Finally, if $r \in R$, and $k \in K$, then $(r, f(r))(0, k) = (0, f(r)k) \in R \bowtie K$, which implies that $f(r)k \in K$. It follows that K is an $f(R)-$subalgebra of R'.

$\square$

Corollary 2.1.2. *Let R be a ring, and let $K \subseteq R$ have the same operations of addition and multiplication defined. Then $R \bowtie K := \{(r, r+k) | r \in R, k \in K\}$ is a ring if and only if K is an ideal of R.*

Note that there was no requirement in these lemmas that K be a proper subset (or proper ideal) or nonzero. The proofs are still valid, considering the rings $R \bowtie^f 0 \cong R$ and $R \bowtie^f R' = R \times R'$ (or $R \bowtie 0 \cong R$ and $R \bowtie R \cong R \times R$, respectively in the "simple bowtie ring" case).

Lemma 2.1.3. *Let $f : R \to R'$ be a homomorphism of rings, and let H, K be $f(R)-$ subalgebras of R'. Then $R \bowtie^f H = R \bowtie^f K$ if and only if $H = K$.*

Proof. If $H = K$ then clearly $R \bowtie^f H = R \bowtie^f K$. For the converse note that if there is an element $h \in H \setminus K$ then $(0, h) \in R \bowtie^f H \setminus R \bowtie^f K$.

$\square$

We should point out that nothing in this lemma prevents two bowtie rings $R \bowtie^f H$, $R \bowtie^f K$ (for distinct H, K) from being isomorphic to one another. This is certainly possible, but we are only concerned with whether or not the two bowtie rings are equal as sets.

The next lemma will be vital in our study of many properties of ring extensions in the Integrality chapter. Also, a version concerning general bowtie rings will be presented and used briefly in our study of integrally closed minimal extensions in the Flat Epimorphisms chapter. For the sake of clarity, we present the lemma in the context of simple bowtie rings here. In the proof, recall that if $R \subset T$ is a ring extension, and $Q \in Spec(T)$, then the contraction of Q to R is a prime ideal of R (that is, $Q \cap R \in Spec(R)$).

Lemma 2.1.4. *Let $R \subset T$ be a ring extension with ideals I, J, respectively, such that $I \subseteq J$. Suppose that Q' is a prime ideal of $T \bowtie J$. If Q' has the form $\{(q, q+j) | q \in Q, j \in J\}$ (resp. $\{(q+j, q) | q \in Q, j \in J\}$) for some prime Q of T, then $Q' \cap (R \bowtie I) =$*

$\{(p,p+i) | p \in Q \cap R, i \in I\}$ (resp. $\{(p+i,p) | p \in Q \cap R, i \in I\}$). Thus, Q' and $Q' \cap (R \bowtie I)$ can be written in the same form (in the sense of Lemma 1.5.8).

Proof. First consider the case $Q' = \{(q,q+j) | q \in Q, j \in J\}$. Let $P' = \{(p,p+i) | p \in Q \cap R, i \in I\}$. Then P' is a prime ideal in $R \bowtie I$ by Lemma 1.5.8 and clearly $P' \subseteq Q'$. Thus $P' \subseteq Q' \cap (R \bowtie I)$.

For the reverse inclusion, let $(q,q+j) \in Q'$. If $(q,q+j) \in R \bowtie I$ as well, then this element must have the form $(r,r+i)$ for an $r \in R$ and an $i \in I$. Then since $(q,q+j) = (r,r+i)$, clearly we must have $r = q$. It follows that $q \in R$, so that $q \in Q \cap R$. Further, since $(q,q+j) = (r,r+i) = (q,q+i) \in P'$, we have the reverse inclusion that we desired.

Next we consider the only other possible case (by Lemma 1.5.8), viz., $Q' = \{(q+j,q) | q \in Q, j \in J\}$. Let $P' = \{(p+i,p) | p \in Q \cap R, i \in I\}$ As before we see that $P' \subseteq Q' \cap (R \bowtie I)$.

For the reverse inclusion, let $(q+j,q) \in Q'$. If $(q+j,q) \in R \bowtie I \subseteq R \times R$ as well, then $q \in R$ so that $q \in Q \cap R$. Further, by definition of $R \bowtie I$, the difference $(q+j) - q = j$ must be an element of I. Therefore $(q+j,q) \in P'$, and our reverse inclusion follows.

$\square$

2.2 Constructions and Decompositions

In this section we will examine sums, composites, and intersections of bowtie rings; these lemmas will be particularly useful later in our study of intermediate rings. For a few small immediate applications, we consider when arbitrary rings can be obtained by performing one of these operations on a finite collection of non-trivial simple bowtie rings. By a non-trivial simple bowtie ring, we mean one of the form $R \bowtie I$ with $I \neq R$ a nonzero ideal of R; the "nonzero" requirement in our decompositions is necessary, since if $I = 0$ we trivially have that $R \cong R \bowtie I$.

Lemma 2.2.1. *Let R, S be subrings of some ambient ring A, and R', S' be subrings of some ambient ring A'. Suppose further that $R \cap S$ and $R' \cap S'$ are rings. Let $f : R \to R', g : S \to S'$*

be ring homomorphisms that agree on $R \cap S$, let I be an $f(R)-$subalgebra of R' and let J be a $g(S)-$subalgebra of S'. Then $(R \bowtie^f I) \cap (S \bowtie^g J) = (R \cap S) \bowtie^f (I \cap J)$. In particular, setting $R = S$ and $R' = S'$, then $(R \bowtie^f I) \cap (R \bowtie^f J) = R \bowtie^f (I \cap J)$.

Proof. We easily note that $(R \bowtie^f I) \cap (S \bowtie^g J) = \{(r, f(r) + i) | r \in R, i \in I\} \cap \{(s, g(s) + j) | s \in S, j \in J\} = \{(t, f(t) + k) | t \in R \cap S, k \in I \cap J\} = (R \cap S) \bowtie^f (I \cap J)$.

□

Corollary 2.2.2. *Every ring can be written (isomorphically) as the intersection of two nontrivial simple bowtie rings.*

Proof. Let R be any ring and let X, Y be two distinct indeterminates over R. Note that $R[X]$ and $R[Y]$ are subrings of the ambient ring $R[X, Y]$. Since $R[X]$ contains no multiples of Y (and $R[Y]$ contains no multiples of X), it follows that the only elements shared by the subrings $R[X]$ and $R[Y]$ are the constant terms, i.e., $R[X] \cap R[Y] = R$. Similarly, the ideal generated by X in $R[X]$ contains no Y terms (even though the ideal generated by X in $R[X, Y]$ does), and the ideal $YR[Y]$ contains no X terms; hence $XR[X] \cap YR[Y] = 0$.

Now set $S := R[X] \bowtie XR[X]$ and $S' := R[Y] \bowtie YR[Y]$, and note that these are both contained as subrings of the ambient ring $R[X, Y] \bowtie (X, Y)R[X, Y]$. Then by Lemma 2.2.1, $S \cap S' = (R[X] \cap R[Y]) \bowtie XR[X] \cap YR[Y] = R \bowtie 0 \cong R$.

□

Corollary 2.2.3. *Let R be a Noetherian ring. Then any simple bowtie ring $R \bowtie I$ can be written as the finite intersection of simple bowtie extensions $R \bowtie P_1, R \bowtie P_2, ..., R \bowtie P_n$ of R where the P_i are primary ideals of R. In particular, any Noetherian ring R in which 0 is not primary can be represented as (an isomorphic copy of) the finite intersection of nontrivial simple bowtie extensions of R.*

Proof. Allowing the possibility that $P_i = 0$ or $P_i = I$ for the primary ideals P_i, the first statement follows easily from the Lasker-Noether Theorem and Lemma 2.2.1. For the second statement, let $P_1 \cap \cdots \cap P_n$ be a primary decomposition for 0. Then $R \cong R \bowtie 0 = R \bowtie P_1 \cap \cdots \cap R \bowtie P_n$.

□

Corollary 2.2.4. *No domain R can be written (isomorphically) as a finite intersection of nontrivial simple bowtie extensions of R.*

Proof. Suppose that $R \cong (R \bowtie I_1) \cap \cdots \cap (R \bowtie I_n)$ for some nonzero ideals $I_1, ..., I_n$ of R. Then by Lemmas 2.2.1 and 2.1.3, and the fact that $R \cong R \bowtie 0$ we have that $I_1 \cap \cdots \cap I_n = 0$, so that in particular, for any nonzero elements $i_1, ..., i_n$ of $I_1, ..., I_n$, respectively, we have the product $i_1 \cdots i_n = 0$. But since each of the elements i_k was nonzero by assumption, this contradicts R being a domain.

□

However, when we move to infinite intersections, we have the contrary result.

Proposition 2.2.5. *Every Noetherian domain R that is not a field can be represented (isomorphically) as an infinite intersection of nontrivial simple bowtie extensions of R.*

Proof. Suppose that R is a Noetherian domain that is not a field, and let I be any nonzero ideal of R. Note that I is not nilpotent since R is a domain. Thus for each natural number n, $R \bowtie I^n$ is a nontrivial bowtie extension of R. Clearly the proof of Lemma 2.2.1 extends to infinite intersections. Then we have by the Krull Intersection Theorem, $\bigcap(R \bowtie I^n) = R \bowtie \bigcap I^n = R \bowtie 0 \cong R$.

□

The proof of the following lemma is straightforward, and similar to that of Lemma 2.2.1. This lemma will become useful when we study lattices of subrings in the following chapter.

Lemma 2.2.6. *Let $f : R \to R'$ be a ring homomorphism, and let I, J be $f(R)-$subalgebras of R'. Then $(R \bowtie^f I) + (R \bowtie^f J) = R \bowtie^f (I + J)$ (where the first sum is taken in $R \times R$).*

Recall that by the *composite* of two subrings R, S of some ring A, we mean the subring $RS := \{\sum_{i=1}^{n} r_i s_i | r_i \in R, s_i \in S, n \in \mathbb{N}\}$ of A.

Lemma 2.2.7. *Let I, J be ideals of rings R, S, respectively. Assume that the composite RS exists and that the sum $SI + IJ + RJ$ exists (as an $RS-$algebra). Then $(R \bowtie I)(S \bowtie J) = RS \bowtie (SI + IJ + RJ)$. In particular, if I, J are ideals of the same ring R, then $(R \bowtie I)(R \bowtie J) = R \bowtie (I + J)$.*

Proof. To simplify notation we will not write the index terms on finite sums in this proof. Any element of $(R \bowtie I)(S \bowtie J)$ can be written as

$$\sum (r_k, r_k + i_k)(s_k, s_k + j_k) = (\sum (r_k s_k), \sum (r_k s_k + r_k j_k + s_k i_k + i_k j_k)) =$$

$$(\sum r_k s_k, \sum r_k s_k + \sum r_k j_k + \sum i_k j_k + \sum s_k i_k),$$

whence $(R \bowtie I)(S \bowtie J) \subseteq RS \bowtie (SI + IJ + RJ)$.

For the reverse inclusion, let $(\sum r_k s_k, \sum r_k s_k + \sum s_h i_h + \sum i_l j_l + \sum r_t j_t)$ be an arbitrary element of $RS \bowtie (SI + IJ + RJ)$. This element can be rewritten as $\sum (r_k, r_k)(s_k, s_k) + \sum (0, i_h)(s_h, s_h) + \sum (0, i_l)(0, j_l) + \sum (r_t, r_t)(0, j_t)$, so we see that it lies in $(R \bowtie I)(S \bowtie J)$, as $(r_k, r_k), (0, i_h), (0, i_l), (r_t, r_t) \in R \bowtie I$ and $(s_k, s_k), (s_h, s_h), (0, j_l), (0, j_t) \in S \bowtie J$.

$\square$

Corollary 2.2.8. *Every finite composite of nontrivial simple bowtie extensions of a quasilocal ring (R, m) can be written as a single nontrivial simple bowtie extension of R.*

Proof. For any proper ideals $I_1, ..., I_n$ of R, we have $(R \bowtie I_1) \cdots (R \bowtie I_n) = R \bowtie (I_1 + \cdots + I_n)$. Since all of the I_k are nonzero by assumption, clearly their sum is nonzero as well, and since $I_1 + \cdots + I_n \subseteq m$, we know that the sum of these ideals is not all of R (thus the bowtie ring is nontrivial).

$\square$

Corollary 2.2.9. *No domain R can be written (isomorphically) as a finite composite of nontrivial simple bowtie extensions of R.*

Proof. Suppose that R could be written as the finite composite of nontrivial simple bowtie extensions of R. Then, as in the first corollary, R can be written (isomorphically) as a simple bowtie ring $R \bowtie I$ for some (now possibly improper) ideal I of R. It follows from our assumption (and the proof of Lemma 2.2.7) that I is nontrivial, say $0 \neq i \in I$. Then R is not a domain, since it contains the nonzero elements $(0, i), (i, 0)$, whose product is zero. $\square$

Note that if M, N are comaximal ideals of a ring R (i.e., $M+N = R$) then $(R \bowtie M)(R \bowtie N) = R \bowtie (M + N) = R \bowtie R = R \times R$. However if (R, m) is quasilocal, then for any two proper ideals I, J of R, we see that $(R \bowtie I)(R \bowtie J) = R \bowtie (I + J) \subseteq R \bowtie m \subset R \times R$. Since any two distinct maximal ideals are comaximal, we now have the following:

Corollary 2.2.10. *Let R be a ring. Then $R \times R$ can be written as a finite composite of nontrivial simple bowtie extensions of R if and only if R is not quasilocal.*

Chapter 3: Intermediate Rings and Minimal Ring Extensions

3.1 Introduction and Definitions

Given an extension of bowtie rings $R \bowtie^f I \subset T \bowtie^f J$ (in the sense of Remark 1.4.1) we wish to investigate the intermediate rings of this extension. In this chapter we will determine the possible forms of such rings, and study the cardinality of existing chains of intermediate rings. We will also determine under what conditions the extension $R \bowtie^f I \subset T \bowtie^f J$ is minimal.

Given a ring extension $R \subseteq T$ we will adopt the notation $[R,T]$ for the set (partially ordered by inclusion) of all rings S satisfying $R \subseteq S \subseteq T$ (as unital subrings). Similarly, for two $R-$subalgebras $I \subseteq J$ of an $R-$algebra A, we will define $[I,J]$ as the set of all $R-$subalgebras K of A satisfying $I \subseteq K \subseteq J$ (naturally, if $R = A$ then I, J are ideals of A and $[I,J]$ refers to all the ideals K of A with $I \subseteq K \subseteq J$).

As in [GH], we will call a ring extension $R \subset S$ a $\Delta-$*extension* if for any two intermediate rings $R_1, R_2 \in [R,S]$, the sum $R_1 + R_2$ is a subring of S. We will follow [G] in defining a ring extension $R \subseteq T$ as a $\lambda-$*extension* if the set $[R,T]$ is linearly ordered (by inclusion). Note that every $\lambda-$extension $R \subset T$ is also a $\Delta-$extension, since any two intermediate rings A, B are comparable, say $A \subseteq B$, so their sum $A + B = B$ is a ring. For an example of a $\Delta-$extension that is not a $\lambda-$extension, see Example 3.2.9. A ring whose ideals are linearly ordered by inclusion is called a *chained ring* (thus a chained ring with no nonzero zero-divisors is a valuation domain). An extension $R \subset T$ satisfies *FIP* (or the *finitely many intermediate algebras property*) if $[R,T]$ is a finite set and satisfies *FCP* (or the *finite chain property*) if every chain of intermediate rings is finite. Clearly any extension satisfying FIP must satisfy FCP as well.

3.2 Intermediate Rings

We begin by noting the exact form of any intermediate ring in the set $[R \bowtie^f I, R \bowtie^f J]$. In particular we will prove that not only is every simple bowtie ring an element of the set $[R^{\Delta}, R \times R]$, but every element of this set is a simple bowtie ring. Thus the set of all simple bowtie extensions of a ring R is exactly the set of all rings between R (embedded along the diagonal) and $R \times R$. The proof is very similar to an argument in [D,2 Remark 2.9] that every $R-$subalgebra of an idealization $R \ltimes E$ has the form $R \ltimes E'$ for some $R-$submodule E' of the $R-$module E.

Lemma 3.2.1. *Let $f : R \to R'$ be a ring homomorphism, and let $I \subset J$ be $f(R)-$subalgebras of R'. If S is a ring with $R \bowtie^f I \subset S \subset R \bowtie^f J$, then $S = R \bowtie^f K$ for some $f(R)-$subalgebra K of R' with $I \subset K \subset J$.*

Proof. Fix an intermediate ring $R \bowtie^f I \subset S \subset R \bowtie^f J$. Note that each element of S has the form $(r, f(r) + j)$ for some $r \in R$, $j \in J$. Now define the set $K := \{j \in J | \exists r \in R \text{ with } (r, f(r) + j) \in S\}$. We claim $S = R \bowtie^f K := \{(r, f(r) + k) | k \in K\}$.

Given $(r, f(r) + k) \in R \bowtie^f K$ then for some $r' \in R$, $(r', f(r') + k) \in S$. Note that $(r', f(r')) \in R \bowtie^f I \subset S$ so we have that $(0, k) \in S$. Further, $(r, f(r)) \in S$ so that $(r, f(r) + k) \in S$; thus $R \bowtie^f K \subseteq S$. If $(r, f(r) + j) \in S$, then $j \in K$ by definition of K. It follows that $S \subseteq R \bowtie^f K$ so that these sets are equal. Now the fact that K is an $f(R)-$subalgebra of R' follows from Lemma 2.1.1. Finally, the inclusions $I \subset K \subset J$ follow from the definition of S and the fact that $R \bowtie^f I \subset S = R \bowtie^f K \subset R \bowtie^f J$.

$\square$

Let $f : R \to R'$ be a homomorphism of rings. For any $f(R)-$subalgebra I of R' the bowtie ring $R \bowtie^f I$ lies in $[\Gamma(f), R \times R']$ by construction. Since $\Gamma(f) = R \bowtie^f 0$ and $R \times R' = R \bowtie^f R'$, the lemma tells us that $[\Gamma(f), R \times R']$ only consists of such rings. Thus we have the following.

Corollary 3.2.2. *Let $f : R \to R'$ be a ring homomorphism. Then the set of all rings $R \bowtie^f I$ for I an (possibly trivial or improper) $f(R)-$subalgebra of R' is equal to $[\Gamma(f), R \times R']$. In particular the set of all simple bowtie rings $R \bowtie I$ (for a possibly trivial or improper ideal I of R) is equal to $[R^\Delta, R \times R]$.*

Theorem 3.2.3. *Let $f : R \to R'$ be a ring homomorphism, and let $I \subset J$ be $f(R)-$subalgebras of R'. Then there is a one-to-one, order-preserving correspondence mapping the set $[R \bowtie^f I, R \bowtie^f J]$ onto the set $[I, J]$, given by the map $R \bowtie^f K \mapsto K$.*

Proof. By Lemma 2.1.3 the given map is well-defined and injective. From Lemma 3.2.1 the map preserves order. Finally, surjectivity is obvious since $R \bowtie^f K$ is a ring between $R \bowtie^f I$ and $R \bowtie^f J$ for any $f(R)-$subalgebra K between I and J.

□

In particular, if $I \subset J$ are ideals of a ring R, then there is a one-to-one, order-preserving correspondence between the rings in $[R \bowtie I, R \bowtie J]$ and the ideals in $[I, J]$. This consequence has in fact already been presented in a noncommutative context in [DM, Lemma 2.4], but only in the special case where $I = 0$.

We record a few more consequences together in the following corollary. As these individual results will not be much use to us, the main purpose of this corollary is to note that our Theorem 3.2.3 in fact generalizes some other recently presented results (see Corollary 3.5.4 and the comments preceding it).

Corollary 3.2.4. *Let $f : R \to R'$ be a homomorphism of rings, and let $I \subseteq J$ be $f(R)-$subalgebras of R'. Identify R with $\Gamma(f) := R \bowtie^f 0 (\cong R)$. Then*

- *$R \subset R \bowtie^f R'$ has FCP if and only if every chain of $f(R)-$subalgebras of R' is finite.*
- *$R \subset R \bowtie^f I$ has FIP if and only if I has only finitely many $f(R)-$subalgebras.*
- *$R \subset R \bowtie^f I$ has FCP if and only if every chain of $f(R)-$subalgebras of I is finite.*

- *$R \bowtie^f I \subset R \bowtie^f J$ has FIP if and only if the $f(R)-$algebra J/I has only finitely many $f(R)-$subalgebras.*

- *$R \bowtie^f I \subset R \bowtie^f J$ has FCP if and only if every chain of $f(R)-$subalgebras of the $f(R)-$algebra J/I is finite.*

Proof. The first three statements are immediate. For the remaining two statements, note that the set of $f(R)-$subalgebras of J/I is the set of $f(R)-$algebras of the form K/I for an intermediate $f(R)-$algebra $I \subseteq K \subseteq J$ of R.

□

Corollary 3.2.5. *Let $f : R \to R'$ be a ring homomorphism, and let $I \subset J$ be $f(R)-$subalgebras of R'. Then $R \bowtie^f I \subset R \bowtie^f J$ is a minimal ring extension if and only if J/I is a simple $f(R)-$submodule of R'/I.*

Corollary 3.2.6. *Given two ideals $I \subset J$ of a ring R, the extension $R \bowtie I \subset R \bowtie J$ is a $\Delta-$extension. Thus for any ring R, the extension $R^{\Delta} \subset R \times R$ is a $\Delta-$extension.*

Proof. By Lemma 3.2.1, every intermediate ring in $[R \bowtie I, R \bowtie J]$ is a bowtie ring of the form $R \bowtie K$ for some ideal K of R. By Lemma 2.2.6 the sum of any two of these is again a ring.

The last statement is now immediate, setting $I = 0, J = R$ and recalling that $R^{\Delta} = R \bowtie 0$ and $R \times R = R \bowtie R$.

□

As in [G] we will refer to an integral domain R as a $\lambda-$*domain* if $R \subseteq Frac(R)$ is a $\lambda-$extension.

Corollary 3.2.7. *Let R be a ring. Then R is a chained ring if and only if $R \bowtie I \subset R \bowtie J$ is a $\lambda-$extension for all ideals $I \subset J$ of R. If R is a domain, then these conditions are equivalent to R being a $\lambda-$domain.*

Proof. The first claim follows directly from Theorem 3.2.3. For the final statement, note that any chained ring that is a domain is a valuation domain, which is integrally closed by [K, Theorem 50]. The statement now follows from [G, Corollary 1.5]. $\square$

We note that the following corollary to Corollary 3.2.7 was also proved via different methods in [G, Corollary 2.15(a)].

Corollary 3.2.8. *Let R be a ring. Then $R^{\Delta} \subset R \times R$ is a $\lambda-$extension if and only if R is a chained ring.*

Example 3.2.9. In light of Corollaries 3.2.6-3.2.8, we can now easily give an example of a $\Delta-$extension that is not a $\lambda-$extension. Simply pick any non-chained ring R (for instance $R = \mathbb{Z}$), and take the extension $R^{\Delta} \subset R \times R$.

For an example involving nontrivial simple bowtie rings, set $R = \mathbb{Q}[X, Y]$, $I = (XY)$, $J = (X, Y)$. Then $R \bowtie I \subset R \bowtie J$ is a $\Delta-$extension. However it is not a $\lambda-$extension, since by Theorem 3.2.3 the set $[R \bowtie I, R \bowtie J]$ contains the incomparable rings $R \bowtie (X)$ and $R \bowtie (Y)$.

In the following proof recall that every Artinian ring is also necessarily Noetherian (cf. [M, Theorem 3.2]).

Corollary 3.2.10. *Let R be a ring. Then R is Artinian if and only if $R \bowtie I \subset R \bowtie J$ satisfies FCP for all ideals $I \subset J$ of R.*

Proof. We prove each direction via contrapositive. Suppose there are two ideals $I \subset J$ of R such that the ring extension $R \bowtie I \subset R \bowtie J$ does not satisfy FCP. Then there is an infinite chain (which we can harmlessly assume is countable) of rings $R \bowtie I \subset \cdots \subset S_{-1} \subset S_0 \subset S_1 \subset \cdots \subset R \bowtie J$. By (a simple iteration of) Lemma 3.2.1, each S_i has the form $R \bowtie K_i$ for an ideal K_i of R, with $K_i \subset K_{i+1} \subset K_{i+2} \cdots$. Fix the intermediate ring S_0. By assumption at least one of the chains of rings $S_0 \supset S_{-1} \supset S_{-2} \cdots$ or $S_0 \subset S_1 \subset S_2 \cdots$ does not terminate. In the first case we would have that $K_0 \supset K_{-1} \supset K_{-2} \cdots$ is a descending chain of ideals that does not terminate, giving that R is not Artinian. In the second case the ascending chain of ideals $K_0 \subset K_1 \subset K_2 \cdots$ does not terminate, giving that R is not Noetherian and so not Artinian either.

For the converse, assume that R is not Artinian and let $I_0 \supset I_1 \supset I_2 \supset \cdots$ be a descending chain of ideals that does not terminate. Set $J = I_0$ and $I = \bigcap I_k$. Then $R \bowtie I \subset R \bowtie J$ does not satisfy FCP since the within this extension lies the chain of intermediate rings $\cdots \subset R \bowtie I_2 \subset R \bowtie I_1 \subset R \bowtie I_0$.

□

We will say that a ring extension $R \subset T$ has $FACP$ (or the *finite ascending chain property*), if every ascending chain of intermediate rings $R \subset S_1 \subset S_1 \subset S_3 \subset \cdots$ with each S_i strictly contained in T has a maximal element. We will analogously define $FDCP$ (or the *finite descending chain property*) as each descending chain $T \supset S_1 \supset S_2 \supset S_3 \supset \cdots$ containing a minimal element (now assuming R is strictly contained in each S_i). The Artinian part of this corollary follows easily by a similar proof to Corollary 3.2.10 above (by trimming some unnecessary pieces); the Noetherian part follows via similarly methods, replacing the intersection in the last paragraph, of course, with a union.

Corollary 3.2.11. *Let R be a ring. Then R is Artinian (resp., Noetherian) if and only if $R \bowtie I \subset R \bowtie J$ satisfies FDCP (resp., FACP) for all ideals $I \subset J$ of R.*

It is known that the ideals of a Prufer domain form a distributive lattice (cf. [G2, Theorem 25.2, p. 310]). A ring- possibly not a domain- with this property is called an *arithmetical ring*. With this we can now extend Theorem 3.2.3 from a one-to-one correspondence to a lattice isomorphism.

Theorem 3.2.12. *Let $I \subset J$ be ideals of an arithmetical ring R. Then the set of rings in $[R \bowtie I, R \bowtie J]$ forms a distributive lattice, where the join of two rings S_1, S_2 is $S_1 \vee S_2 := S_1 + S_2$. Moreover, there is a lattice isomorphism between the rings in $[R \bowtie I, R \bowtie J]$ and the ideals in $[I, J]$ given by $f(R \bowtie K) = K$. Conversely, if such a lattice (and lattice isomorphism) exists for every pair of ideals $I \subset J$ in a ring R, then R must be arithmetical.*

Proof. Suppose that R is an arithmetical ring, so that its ideals form a distributive lattice, where the join and meet of two ideals I_1, I_2 are defined as $I_1 + I_2$ and $I_1 \cap I_2$, respectively. By Lemma 3.2.1, every ring in $[R \bowtie I, R \bowtie J]$ can be written as $R \bowtie K$ for some ideal K of R, and by Lemmas 2.2.1, 2.2.7, and 2.2.6 we conclude that the rings in $[R \bowtie I, R \bowtie J]$ form a lattice (where the join of two subrings $R \bowtie H, R \bowtie K$ of $R \bowtie J$ is the smallest subring containing them, namely the composite $(R \bowtie H)(R \bowtie K) = R \bowtie (H + K) = R \bowtie H + R \bowtie K$). This lattice is distributive since for any ideals $H, K, L \in [I, J]$ of the arithmetical ring R we have

$$(R \bowtie H) \cap (R \bowtie K + R \bowtie L) = (R \bowtie H) \cap (R \bowtie (K + L)) = R \bowtie (H \cap (K + L)) =$$

$$R \bowtie ((H \cap K) + (H \cap L)) = R \bowtie (H \cap K) + R \bowtie (H \cap L) =$$

$$(R \bowtie H) \cap (R \bowtie K) + (R \bowtie H) \cap (R \bowtie L).$$

Now define the map $f : [R \bowtie I, R \bowtie J] \to [I, J]$ by $f(R \bowtie K) = K$. By Theorem 3.2.3 this map is bijective and preserves inclusion. We now note that it also preserves the minimum lattice element, as $f(R \bowtie I) = I$, and the maximum lattice element, as $f(R \bowtie J) = J$. We know by Lemma 2.2.1 that meets are preserved, and it follows by Lemma 2.2.7 that joins are preserved (recall that $H + K$ is the smallest ideal containing the ideals H, K of a given ring, and thus (in an arithmetical ring) is the join of H and K in the lattice of ideals of the ring). Therefore f is a lattice isomorphism.

For the final statement, suppose that R is not arithmetical, and thus there exist ideals H, K, L of R where $H \cap (K + L) \neq (H \cap K) + (H \cap L)$. Then, following similar reasoning as above (and recalling Lemma 2.1.3), we see that

$$(R \bowtie H) \cap (R \bowtie K + R \bowtie L) = R \bowtie (H \cap (K + L)) \neq$$

$$R \bowtie ((H \cap K) + (H \cap L)) = (R \bowtie H) \cap (R \bowtie K) + (R \bowtie H) \cap (R \bowtie L),$$

so that the rings in $[R \bowtie 0, R \bowtie (H + K + L)]$ do not form a distributive lattice, and so in particular, constructing the described lattice isomorphism is impossible.

$\square$

Next we want to investigate extensions of bowtie rings where the $f(R)-$algebra J (rather than the ring R) is kept fixed.

Lemma 3.2.13. *Let $R \subset T$ be an extension of rings, let $f : T \to T'$ be a ring homomorphism, and let J be an $f(T)-$subalgebra of T'. If $R \bowtie^f J \subset A \subset T \bowtie^f J$ for some ring A, then A has the form $A = S \bowtie^f J$ for some ring S with $R \subset S \subset T$.*

Proof. Elements of A naturally look like $(t, f(t) + j)$ where $t \in T$ and $j \in J$. Let $S = \{t \in T | \exists j \in J \text{ with } (t, f(t) + j) \in A\}$. Clearly $R \subseteq S \subseteq T$, and assuming S is a ring we claim that $A = S \bowtie^f J$. Say $(t, f(t) + j) \in A$. Then $t \in S$ by definition, so it follows that $A \subseteq S \bowtie^f J$. Now let $(s, f(s) + j) \in S \bowtie^f J$. Then there exists a $j' \in J$ such that $(s, f(s) + j') \in A$. Since A (containing $R \bowtie^f J$) also contains the set $\{(0, j) | j \in J\}$ we have that $(s, f(s) + j) = (s, f(s) + j') + (0, j) - (0, j') \in A$. Thus $A = S \bowtie^f J$.

To finish the proof, we still need to show that S is a ring. Note that $0 \in S$ since $R \subseteq S$ and that S contains the common 1 to R and T. Let $x, y \in S$. Then $(x, f(x) + i) \in A$ and $(y, f(y) + j) \in A$ for some elements $i, j \in J$. Since A is a ring, $(x - y, f(x - y) + i - j) = (x, f(x)+i)-(y, f(y)+j) \in A$. Then since $i-j \in J$, this gives us $x-y \in S$ by definition of S. Similarly, we have the product $(x, f(x)+i)(y, f(y)+j) = (xy, f(xy)+f(x)j+f(y)i+ij) \in A$ and $f(x)j + f(y)i + ij$ is in J (since J is an $f(R)-$subalgebra of T), so that $xy \in S$, again by definition of S. It follows that S is a ring.

Finally, note that the inclusions $R \subset S \subset T$ must be strict, since otherwise we would have that $A = R \bowtie^f J$ or $A = T \bowtie^f J$.

$\square$

Theorem 3.2.14. *Let $R \subset T$ be a ring extension, let $f : T \to T'$ be a homomorphism of rings, and let J be an $f(T)-$subalgebra of T. Then there is an order-isomorphism between* *the set* $[R \bowtie^f J, T \bowtie^f J]$ *and the set* $[R,T]$*, given by the map* $S \bowtie^f J \mapsto S$*.*

Corollary 3.2.15. *Let* $R \subset T$ *be a ring extension, let* $f : T \to T'$ *be a homomorphism of rings, and let* J *be an* $f(T)-$*subalgebra of* T*. Then* $R \bowtie^f J \subset T \bowtie^f J$ *is a minimal ring extension if and only if* $R \subset T$ *is a minimal extension.*

Corollary 3.2.16. *Let* $R \subset T$ *be a ring extension, let* $f : T \to T'$ *be a homomorphism of rings, and let* J *be an* $f(T)-$*subalgebra of* T*. Then the extension* $R \bowtie^f J \subseteq T \bowtie^f J$ *has FIP (resp., FCP) if and only if the extension* $R \subseteq T$ *has FIP (resp., FCP).*

Corollary 3.2.17. *Let* $R \subset T$ *be a ring extension, let* $f : T \to T'$ *be a homomorphism of rings, and let* J *be an* $f(T)-$*subalgebra of* T*. Then* $R \bowtie^f J \subset T \bowtie^f J$ *is a* $\lambda-$*extension if and only if* $R \subset T$ *is a* $\lambda-$*extension.*

Proposition 3.2.18. *Let* $R \subset T$ *be a ring extension, let* $f : T \to T'$ *be a homomorphism of rings, and let* J *be an* $f(T)-$*subalgebra of* T*. Then* $R \bowtie^f J \subset T \bowtie^f J$ *is a* $\Delta-$*extension if and only if* $R \subset T$ *is a* $\Delta-$*extension.*

Proof. Assume $R \subset T$ is a $\Delta-$extension. Let $A_1, A_2 \in [R \bowtie^f J, T \bowtie^f J]$. By Lemma 4.5 we can write $A_1 = S_1 \bowtie^f J$ and $A_2 = S_2 \bowtie^f J$ for some rings $S_1, S_2 \in [R,T]$. Note that

$$A_1 + A_2 = \{(s_1, f(s_1) + j_1) | s_1 \in S_1, j_1 \in J\} + \{(s_2, f(s_2) + j_2) | s_2 \in S_2, j_2 \in J\} =$$

$$\{(s_1 + s_2, f(s_1) + f(s_2) + j_1 + j_2) | s_1 \in S_1, s_2 \in S_2, j_1, j_2 \in J\} =$$

$$\{(s_1 + s_2, f(s_1 + s_2) + j) | s_1 \in S_1, s_2 \in S_2, j \in J\},$$

which is a ring, since $S_1 + S_2$ is a ring by assumption, and since J is an $f(T)-$subalgebra of $S_1 + S_2$ (being an $f(T)-$subalgebra in both S_1 and S_2).

Conversely, assume $R \bowtie^f J \subset T \bowtie^f J$ is a $\Delta-$extension. If $R \subset T$ is not a $\Delta-$extension then we can find $S_1, S_2 \in [R,T]$ such that $S_1 + S_2$ is not a ring, that is, we can find $s_1, s_1' \in S_1, s_2, s_2' \in S_2$ such that $(s_1 + s_2)(s_1' + s_2\prime) \notin S_1 + S_2$. Note as above that J is an

$f(T)-$subalgebra of S_1+S_2. But then $(s_1+s_2, f(s_1+s_2)), (s_1'+s_2', f(s_1'+s_2'))$ are elements of the set $S_1 \bowtie^f J + S_2 \bowtie^f J$ whose product is not in this set, so that it cannot be a ring. Since $S_1 \bowtie^f J, S_2 \bowtie^f J \in [R \bowtie^f J, T \bowtie^f J]$, this contradicts that $R \bowtie^f J \subset T \bowtie^f J$ is a $\Delta-$extension.

□

Some properties involving intermediate rings descend easily (from the extension $R \bowtie^f I \subset T \bowtie^f J$ to the extension $R \subseteq T$) even while reverse implication may not hold. We will show the "descent" for FCP, FIP, and $\lambda-$extensions in the following two propositions. The "ascent" direction does not hold in either case, as we will show in the next section.

Proposition 3.2.19. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, I an $f(R)-$subalgebra of T', and J an $f(T)-$subalgebra of T' with $I \subseteq J$. If $R \bowtie^f I \subset T \bowtie^f J$ has FCP (resp. FIP), then so does $R \subseteq T$.*

Proof. For any ring S with $R \subseteq S \subseteq T$ we note that $S \bowtie^f J$ is a ring by Corollary 2.1.2. Further, $R \bowtie^f I \subseteq S \bowtie^f J \subseteq T \bowtie^f J$. Thus there is a surjection from the rings in $[R \bowtie^f I, T \bowtie^f J]$ to the rings in $[R, T]$ (by projecting to the first coordinate). This proves the statement for FIP. To prove FCP simply note that the given surjection preserves order (by inclusion).

□

Proposition 3.2.20. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, I an $f(R)-$subalgebra of T', and J an $f(T)-$subalgebra of T' with $I \subseteq J$. If $R \bowtie^f I \subset T \bowtie^f J$ is a $\lambda-$extension, then so is $R \subset T$.*

Proof. Suppose for two rings S_1, S_2 that $R \subset S_1 \subset T$ and $R \subset S_2 \subset T$ with S_1, S_2 incomparable. We will harmlessly abuse notation and write f for f restricted to S_1 (resp. S_2). Then $R \bowtie^f I \subset S_1 \bowtie^f J \subset T \bowtie J$ and $R \bowtie^f I \subset S_2 \bowtie^f J \subset T \bowtie J$. By assumption, we can find an $s_1 \in S_1 \setminus S_2$ and an $s_2 \in S_2 \setminus S_1$. Then $(s_1, f(s_1)) \in S_1 \bowtie J \setminus S_2 \bowtie J$

and $(s_2, f(s_2)) \in S_2 \bowtie J \setminus S_1 \bowtie J$, giving that these two rings are incomparable in $[R \bowtie^f I, T \bowtie^f J]$, so $R \bowtie^f I \subset T \bowtie^f J$ is not a $\lambda-$extension.

$\square$

3.3 Counterexamples

In this section we answer certain questions that may arise due to the results of the previous section. For example, given a homomorphism $f : R \to R'$ with $f(R)-$submodules $I \subseteq J$ of R', if J/I is a simple $R'-$module must $R \bowtie^f I \subset R \bowtie^f J$ be minimal? Must it at least satisfy FCP (or FIP)? Given an extension of simple bowtie rings $R \bowtie I \subset T \bowtie J$ (with ideals $I \subseteq J$ of the rings $R \subseteq T$, respectively) will every intermediate ring be a bowtie ring (in particular, one of the form $S \bowtie K$ where $R \subseteq S \subseteq T$ and $I \subseteq K \subseteq J$)? Also, do the propositions at the end of the preceding section have affirmative converses? Each of these questions is answered in the negative, and we provide appropriate counterexamples in each case (Examples 3.3.5, 3.3.5 (again), 3.3.3 (also 3.3.4), and 3.3.1, respectively).

Example 3.3.1. The converse does not hold for either of the two propositions concluding the previous section. To see this in the first proposition, let $R = T$ be a non-Artinian ring and note by Corollary 3.2.10 that $R \bowtie 0 \subset R \bowtie R$ does not satisfy FCP (or FIP), even though $R \subseteq T = R$ does trivially. For the latter proposition let $R = T$ be any non-chained ring and note that $R \bowtie 0 \subset R \bowtie R$ is not a $\lambda-$extension (Corollary 3.2.8) even though $R \subseteq T = R$ is trivially.

In light of Lemma 3.2.1 and Lemma 3.2.13, it is tempting to assume that any intermediate ring between $R \bowtie^f I \subseteq T \bowtie^f J$ (this extension constructed in the sense of Remark 1.4.1) is a bowtie ring, say $S \bowtie^f K$ for some intermediate ring $S \in [R, T]$ and $f(R)-$algebra $K \in [I, J]$. The following proposition indicates a further necessary condition for this to be true. The subsequent examples show that this condition does not always hold. Given a ring homomorphism $f : R \to R'$ and an $f(R)-$subalgebra I of R', recall that R embeds into $R \bowtie^f I$ via the diagonal $\Gamma(f) := \{(r, f(r)) | r \in R\} \cong R$. Given a subring $S' \subseteq R \times R'$ we use $\pi_1(S')$ to denote the projection of S' onto the first coordinate.

Proposition 3.3.2. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, I an $f(R)-$subalgebra of T', and J an $f(T)-$subalgebra of T' with $I \subseteq J$. Suppose S' is a ring with $R \bowtie^f I \subset S' \subset T \bowtie^f J$. Then $S' = S \bowtie^f K$ for some ring S with $R \subseteq S \subseteq T$ and some $f(S)-$subalgebra K of T' with $I \subseteq K \subseteq J$ if and only if S' contains the diagonal image (namely, $\Gamma(f|_S)$) of $\pi_1(S')$.*

Proof. If $S' = S \bowtie^f K$, then S' contains the diagonal image of S by definition. Conversely, set $S := \pi_1(S')$ (so clearly $R \subseteq S \subseteq T$) and suppose that S' contains the diagonal image S^{Δ^f} of S. Then $S^{\Delta^f} = S \bowtie^f 0 \subseteq S' \subseteq S \bowtie^f J$. If $S' = S \bowtie^f 0$ or $S' = S \bowtie^f J$ we are done (noting that $S' = S \bowtie^f 0$ would force that $I = 0$, so the inclusion $I \subseteq K \subseteq J$ holds). Thus we may assume $S^{\Delta^f} = S \bowtie^f 0 \subset S' \subset S \bowtie^f J$ and we are reduced to the case in Lemma 3.2.1, so that $S' = S \bowtie^f K$ for some $f(S)-$subalgebra K of T' with $0 \subset K \subset J$. Finally, since $R \bowtie^f I \subset S' = S \bowtie^f K$ we can see without much difficulty that $I \subseteq K$.

□

Example 3.3.3. We will use simple bowtie rings for this example. Let $R = \mathbb{Z}$ with ideal $I = 2\mathbb{Z}$, $T = \mathbb{Z}[X]$ with ideal $J = (2, X)$. Note that $R \bowtie I$ is a subring of $T \bowtie J$. We will give an example of a ring in $[R \bowtie I, T \bowtie J]$ that is not of the (bowtie ring) form described in Proposition 3.3.2.

Define $S' := \{(n + 2p(X), n + 2k) | n, k \in \mathbb{Z}, p(X) \in \mathbb{Z}[X]\}$. Direct calculations show that S' is a ring. It is clear that $R \bowtie I \subset S'$. The ring $T \bowtie J$ is the set of all pairs $(f(X), g(X)) \in \mathbb{Z}[X] \times \mathbb{Z}[X]$ where $f(X)$ and $g(X)$ differ by a polynomial with even constant term. Given an element $(n+2p(X), n+2k) \in S'$, the coordinates differ by $2p(X) - 2k$, which obviously has even constant term. It follows that S' is a subring of $T \bowtie J$, so $S' \in [R \bowtie I, T \bowtie J]$.

Note however that S' does not contain the diagonal image of $\pi_1(S')$; for instance, $2X \in \pi_1(S')$, but S' does not contain the element $(2X, 2X)$. Thus S' is not a bowtie extension of $\pi_1(S')$; in particular by Proposition 3.3.2 S' doesn't have the form $S \bowtie K$ for any intermediate ring $R \subseteq S \subseteq T$ and intermediate ideal $I \subseteq K \subseteq J$.

Notice in the previous example that if we reverse the two coordinates in S' then the construction can be viewed as a ring of the form $S \bowtie^f K$ with $R \subseteq S \subseteq T$ and $I \subseteq K \subseteq J$ where K is an $f(R)-$subalgebra of T (namely, with $S = \mathbb{Z}, K = 2\mathbb{Z}[X]$, and $f : \mathbb{Z} \to \mathbb{Z}[X,Y]$ the natural inclusion map). We now present an example where this is not possible.

Example 3.3.4. Consider the rings $R = \mathbb{Z}$ with ideal $I = 2\mathbb{Z}$ and $T = \mathbb{Z}[X,Y]$ with ideal $J = (2, X, Y)$. Then $R \bowtie I \subset T \bowtie J$ is a ring extension. Now consider the set $S' := \{(n + 2p(X), n + 2q(Y)) | n \in \mathbb{Z}, p(X) \in \mathbb{Z}[X], q(Y) \in \mathbb{Z}[Y]\}$. Then S' is a ring, and $R \bowtie I \subset S'$. Now $T \bowtie J$ is the set of all polynomials $(g(X,Y), h(X,Y)) \in \mathbb{Z}[X,Y]$ where $g(X,Y)$ and $h(X,Y)$ differ by a polynomial in $\mathbb{Z}[X,Y]$ with even constant term. Given any element $(n + 2p(X), n + 2q(Y))$ in S', the two polynomials in this pair differ by $n + 2p(X) - n - 2q(Y) = 2(p(X) - q(Y))$, which is clearly a polynomial in $\mathbb{Z}[X,Y]$ with even constant term. It follows that S' is a subring of $T \bowtie J$.

As in the previous example we can see that S' cannot be represented in the form $S \bowtie^f K$ since $\pi_1(S')$ does not embed into S' along the diagonal. However, unlike in the previous example $\pi_2(S')$ does not embed into S' along the diagonal either so that switching the coordinates still does not produce a ring of the desired form $S \bowtie^f K$ with $R \subseteq S \subseteq T$ and $I \subseteq K \subseteq J$.

Let $f : R \to R'$ be a ring homomorphism. Currently the most common construction of a bowtie ring is to take an ideal of R', say I, and construct the ring $R \bowtie^f I$. However, as we have seen, many concepts generalize to when I is an $f(R)-$subalgebra of R'. (Again, in this document we do not make the assumption that algebras and subalgebras necessarily contain a unit element.) Further, in considering intermediate rings we are forced to take such sets into account. Below we give some examples showing that some of the concepts we have applied using algebras would not respond as well to ideals.

Example 3.3.5. *An example where J/I is a simple R'-module but $R \bowtie^f I \subset R \bowtie^f J$ does not satisfy FCP (or FIP), thus is not minimal.*

Set $R = \mathbb{Z}$ and $R' = \mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$. Let $I = \{(q, 0, 0)|q \in \mathbb{Q}\}$ and $J = \{(q_1, 0, q_2)|q_1, q_2 \in \mathbb{Q}\}$. Then I, J are nonzero proper ideals of R' with $I \subset J$. Let $f : R \to R'$ be the natural embedding $n \mapsto (n, n, n)$.

For each $n \in \mathbb{N}$ create the $f(R)-$subalgebra $H_n := \{((q, 0, c)|q \in \mathbb{Q}, c \in 2^n\mathbb{Z})\}$ of R'. Note that the H_n are not ideals in R', though $I \subset H_n \subset J$. In fact, $J/I \cong \mathbb{Q}$ is a simple $R'-$module. However, now in $[R \bowtie^f I, R \bowtie^f J]$ we have the non-terminating chain of intermediate rings $\cdots R \bowtie^f H_3 \subset R \bowtie^f H_2 \subset R \bowtie^f H_1$. Thus the extension $R \bowtie^f I \subset R \bowtie^f J$ does not satisfy FCP (and consequently, does not satisfy FIP either).

In the following example the $R'-$module J/I is not simple. We present this as an example of when the intermediate rings involved may or may not be constructed from an ideal of R'. In fact, there are uncountably many intermediate rings of either type.

Example 3.3.6. Let $R = \mathbb{Z}$ and $R' = \prod_{\mathbb{N}} \mathbb{Q}$. Define $I = \{(q, 0, 0, 0, ...)|q \in \mathbb{Q}\}$, $J = \{(q_1, 0, q_3, q_4, ...)|q_i \in \mathbb{Q}\}$ so $I \subset J$ are ideals of R'. Let $H = \{(q, 0, z_1, z_2, ...)|q \in \mathbb{Q}, z_i \in \mathbb{Z}\}$ and note H is not an ideal of R'. However, letting $f : R \to R'$ be the natural inclusion $f(n) = (n, n, n, ...)$, we see that $R \bowtie^f H$ is a ring and $R \bowtie^f I \subset R \bowtie^f H \subset R \bowtie^f J$.

More generally, fix any nonempty subset Γ of $\mathbb{N} \setminus \{1, 2\}$. Let H be the set of elements in J whose entries are integers at every coordinate in Γ. Then once again, H is not an ideal of R' but $R \bowtie^f H$ is a ring and $R \bowtie^f I \subset R \bowtie^f H \subset R \bowtie^f J$. Note that if we instead define H as the set of elements in J whose entries are identically zero at every coordinate in Γ then H is indeed an ideal of R' with $I \subset H \subset J$ (so that again $R \bowtie^f I \subset R \bowtie^f H \subset R \bowtie^f J$).

It follows that in the lattice of elements of $[R \bowtie^f I, R \bowtie^f J]$, we can find an (uncountably) infinite number of intermediate rings $R \bowtie^f H$ where H is an ideal, as well as an (uncountably) infinite number of intermediate rings $R \bowtie^f H$ where H is not.

3.4 Minimal Ring Extensions

Minimal ring extensions have been subject to extensive study (cf. [CDL], [D2], [DS], [DS2], and [FO]). Here we wish to explore minimal extensions involving bowtie rings in more detail. In light of the "general bowtie ring" counterexamples that closed the previous section we will mostly focus on extensions of simple bowtie rings for the present section.

We know that $R \bowtie I \subset R \bowtie J$ will be minimal if J/I is a simple $R-$module (Corollary 3.2.5). We will give a few other equivalent conditions next. Recall that for a ring R and an $R-$module E, the notation $R \ltimes E$ refers to Nagata's idealization of R with E, i.e. the ring of elements $(r, e) \in R \times E$ under the usual addition but with multiplication defined by $(r, e)(s, f) = (rs, rf + se)$. Note that R embeds into $R \ltimes E$ via the map $r \mapsto (r, 0)$ so we may view $R \subseteq R \ltimes E$ as a ring extension.

Proposition 3.4.1. *Let $I \subset J$ be ideals of a ring R and let $\phi : R \to R/I$ be the canonical map. Define $R^{\Delta^f} := \{(r, \phi(r)) | r \in R\} \cong R$. Then the following are equivalent.*

1. *$R \bowtie I \subset R \bowtie J$ is a minimal ring extension.*
2. *$R^{\Delta^f} \subset R \bowtie^{\phi} (J/I)$ is a minimal ring extension.*
3. *$R \subset R \ltimes (J/I)$ is a minimal ring extension.*
4. *J/I is a simple $R-$module.*

Proof. The equivalence of the third and fourth statement follows from [D2, Theorem 2.4]. The equivalence of the first and fourth statements follows from Corollary 3.2.5. Finally we will show the equivalence of the second and fourth statements. First assume that J/I is a simple $R-$module, and let $R^{\Delta^f} \subset S \subseteq R \bowtie^{\phi} (J/I)$ for some ring S. We will use $\overline{r}$ to denote the canonical image of $r \in R$ in R/I.

Take a $(r, \overline{r} + \overline{j}) \in S \setminus \Gamma(f)$, so in particular $\overline{j}$ is a nonzero element of J/I (i.e., $j \in J \setminus I$). Then $(0, \overline{j}) = (r, \overline{r} + \overline{j}) - (r, \overline{r}) \in S$ and $\overline{j}$ generates J/I, so $S = \{(r, \overline{r} + \overline{j}) | r \in R, j \in J\} = R \bowtie^{\phi} J/I$. It follows that $\Gamma(f) \subset R \bowtie^{\phi} (J/I)$ is minimal.

However, assuming that J/I is not simple, we could take a nonzero $R-$submodule $M \subset J/I$ (necessarily an ideal of R/I), so that $\Gamma(f) \subset R \bowtie^{\phi} M \subset R \bowtie^{\phi} J/I$ would give that the extension $\Gamma(f) \subset R \bowtie^{\phi} (J/I)$ is not minimal.

□

In the Intermediate Rings section of this chapter, we have seen two ways to obtain a minimal extension as an extension of bowtie rings (Corollaries 3.2.5 and 3.2.15). By the following proposition these are the only ways to do so.

Proposition 3.4.2. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, I an $f(R)-$subalgebra of T', and J an $f(T)-$subalgebra of T' with $I \subseteq J$. Then $R \bowtie^f I \subset T \bowtie^f J$ is a minimal ring extension if and only if one of the following holds:*

1. *$R = T$ and J/I is a simple $f(T)-$submodule of T'.*

2. *$R \subset T$ is a minimal ring extension and $J = I$.*

Proof. The cases where $R = T$ or $I = J$ have already been handled, by Corollary 3.2.5 and Corollary 3.2.15, respectively. We wish to show that no other case is possible. But the only remaining possibility to check is $R \subset T$ being a minimal extension, and $I \subset J$ a strict inclusion with no $f(R)-$subalgebras of T lying between I and J. Since J is an $f(R)-$subalgebra of T it is closed under multiplication by elements of T, in particular, by elements of R. Consider the set $R \bowtie^f J := \{(r, f(r) + j) | r \in R, j \in J\}$. It is a quick check to see that this set is a ring (cf. Lemma 2.1.1 and Corollary 2.1.2), and in fact a ring lying strictly between $R \bowtie^f I$ and $T \bowtie^f J$, so that the extension $R \bowtie^f I \subset T \bowtie^f J$ is not minimal.

□

Much study has been devoted to the classification of minimal extensions of certain types of rings. One such classification is given in the following theorem (cf. [PP, pages 369-386]). The subsequent results apply this theorem to extensions of bowtie rings.

Theorem 3.4.3. *Let $A \subset B$ be an integral ring extension. Then $A \subset B$ is a minimal ring extension if and only if there exists $M \in Max(A)$ such that one of the following three conditions holds:*

(a) **inert case***: M is a maximal ideal of B and $A/M \to B/M$ is a minimal field extension;*

(b) **decomposed case***: There exists $q \in B \setminus A$ such that $B = A[q]$, $q^2 - q \in M$, and $Mq \subseteq M$;*

(c) **ramified case***: There exists $q \in B \setminus A$ such that $B = A[q]$, $q^2 \in M$, and $Mq \subseteq M$.*

Furthermore, conditions (a)-(c) are mutually exclusive.

Proposition 3.4.4. *Let $R \subset T$ be an integral ring extension and J an ideal of T (so in particular, $R \bowtie J \subset T \bowtie J$ is an integral extension by the forthcoming Corollary 6.1.2). Then the extension $R \subset T$ is a minimal extension in the inert case (resp., the decomposed case, ramified case) in the sense of Theorem 3.4.3 if and only if $R \bowtie J \subset T \bowtie J$ is a minimal extension of the same case.*

Proof. **Inert case:** Suppose that $R \subset T$ is a minimal extension of the inert case. Then $R \bowtie J \subset T \bowtie J$ is a minimal ring extension by Corollary 3.2.15. By assumption there is a maximal ideal $M \in Max(R)$ such that $M \in Max(T)$ and $R/M \to T/M$ is a minimal field extension. Note that $M' := \{(m, m + j) | m \in M, j \in J\}$ is a maximal ideal of both $R \bowtie J$ and $T \bowtie J$. Further, $(R \bowtie J)/M' \cong R/M$ and $(T \bowtie J)/M' \cong T/M$, by [DFF, Proposition 2.1]. But this clearly implies that $(R \bowtie J)/M' \to (T \bowtie J)/M'$ is a minimal field extension, so that $R \bowtie J \subset T \bowtie J$ is a minimal extension of the inert case.

Conversely, suppose that $R \bowtie J \subset T \bowtie J$ is a minimal extension of the inert case, and let M' be a maximal ideal of $T \bowtie J$, with $(R \bowtie J)/M' \to (T \bowtie J)/M'$ a minimal field extension. We note that the ideal M' has the form $\{(m, m + j) | m \in M, j \in J\}$ or $\{(m + j, m) | m \in M, j \in J\}$ for some $M \in Max(R)$. In the first case we have that

$$(R \bowtie J)/M' = \frac{R \bowtie J}{M \bowtie J} \cong R/M$$

by [DFF, Proposition 2.1]; the second case would follow by a similar homomorphism as used in that proof. Since M' is a maximal ideal of $T \bowtie J$ as well, it must have one of the above forms for some $N \in Max(T)$. If it is of the first form, then clearly $M = N$. Otherwise, we have two ways to write M', shown in the equation $\{(m, m+j) | m \in M, j \in J\} = \{(n+j, n) | n \in N, j \in J\}$. But setting the j to zero (in one side of the equation at a time), it is not difficult to see that, again, $M = N$. In either case, we have that $M \in Max(T)$ and

$$(T \bowtie J)/M' = \frac{T \bowtie J}{R \bowtie J} \cong T/M,$$

so the canonical map $R/M \mapsto T/M$ is just the assumed (minimal) field injection $(R \bowtie J)/M' \to (T \bowtie J)/M'$. It follows that $R \subset T$ is a minimal extension of the inert case.

Decomposed case: Say $R \subset T$ is a minimal extension of the decomposed case. As in the inert case, $R \bowtie J \subset T \bowtie J$ is a minimal extension by Corollary 3.2.15. Then there exists a $q \in T \setminus R$ and an $M \in Max(R)$ with $T = R[q]$, $q^2 - q \in M$, and $Mq \subseteq M$. Set $q' = (q, q)$, and $M' = \{(m, m+j) | m \in M, j \in J\} \in Max(R \bowtie J)$. Note that $q' \in T \bowtie J \setminus R \bowtie J$ so that $T \bowtie J = R \bowtie J[q']$ by minimality. Further, $q'^2 - q' = (q^2 - q, q^2 - q) \in M'$, and $M'q' = \{m'q' | m' \in M'\} = \{(m, m+j)(q, q) | m \in M, j \in J\} = \{(mq, mq + iq) | m \in M, j \in J\} \subseteq M'$ since $mq \in M$ by assumption. It follows that $R \bowtie J \subset T \bowtie J$ is a minimal extension of the decomposed case.

Conversely, suppose that $R \bowtie J \subset T \bowtie J$ is a minimal extension of the decomposed case. Then there exists a $(t, t+j) \in T \bowtie J \setminus R \bowtie J$ and an $M' \in Max(R \bowtie J)$ with $T \bowtie J = R \bowtie J[(t, t+j)]$, $(t, t+j)^2 - (t, t+j) \in M'$ and $M'(t, t+j) \subseteq M'$. Recall that M' must have the form $\{(m, m+j) | m \in M, j \in J\}$ or $\{(m+j, m) | m \in M, j \in J\}$ for some $M \in Max(R)$. For now we will assume that M' is of the first form.

Note that $R \subset T$ is a minimal ring extension (Corollary 3.2.15) and $t \in T \setminus R$, so we must have $T = R[t]$. Further, since $2tj + j^2 - j \in J$ (by definition of the ideal I), we see that $(t, t+j)^2 - (t, t+j) = (t^2 - t, t^2 - t + 2tj + j^2 - j) \in M'$, and so $t^2 - t \in M$. Also,

$M'(t, t + j) = \{(mt, (m + i)(t + j))|m \in M, i \in J\} \subseteq M'$ by assumption, so that $mt \in M$ for all $m \in M$; that is $Mt \subseteq M$.

Finally we consider the possibility that M' be of the second form. Rewrite our given $(t, t + j)$ as $(u + i, u)$ where $u = t + j$ and $i = -j \in J$. Then the remainder of the proof for this case follows a symmetric argument to the one in the previous paragraph (noting- if necessary- that the ring $R \bowtie J$ could harmlessly be defined as $\{(r + j, r)|r \in R, j \in J\}$). It follows that $R \subset T$ is a minimal extension of the decomposed case.

Ramified case: This proof is essentially the same as the decomposed case, noting now that $q^2 \in M$ if and only if $q'^2 = (q^2, q^2) \in M'$.

$\square$

Remark 3.4.5. We know by Proposition 3.4.2 that any minimal extension of the form $T \bowtie J$ of a bowtie ring $R \bowtie I$ has the form $T \bowtie I$ (with $R \subset T$ minimal) or $R \bowtie J$ (where J/I is a simple $R-$module). The first case is integral exactly when $R \subset T$ is (this will be shown in Corollary 6.1.2), and we can classify the extension $R \bowtie I \subset T \bowtie I$ as in Proposition 3.4.4. The second case $R \bowtie I \subset R \bowtie J$ is always integral, as we will see in Corollary 6.1.3, so naturally we ask when it will be of the inert, decomposed, or ramified case. It turns out that this extension will never be of the inert case. By [DFF, Proposition 2.6 (4)] we know that any maximal ideal of $R \bowtie I$ has the form $M' = \{(m, m + i)|m \in M, i \in I\}$ for some $M \in Max(R)$, and any maximal ideal of $R \bowtie J$ has the form $N' = \{(n, n + j)|n \in N, j \in J\}$ for some $N \in Max(R)$. If these two sets are equal, then in particular the set of first coordinates in M' is equal to the set of first coordinates in N' and it follows that $M = N$. Then, since $I \neq J$, it is easy to see that $M' \neq N'$. Thus no maximal ideal of $R \bowtie I$ is a maximal ideal of $R \bowtie J$, so the inert case is impossible.

Now we have that every minimal ring extension $R \bowtie I \subset R \bowtie J$ is of the decomposed case or the ramified case. Either of these cases is possible, and fairly easy to construct. For instance if $I = J^2 \neq J$, then pick any $j \in J \setminus I$ and it is clear that we can take $(0, j)$ as our element q in the ramified case. We can give an example of the decomposed case similarly.

It is not difficult to construct ideals $I \subset J$ of a ring R such that some $j \in J \setminus I$ satisfies $j^2 - j \in I$ (and take $(0, j)$ as our q in the decomposed case). For instance, consider $I = 6\mathbb{Z}$, $J = 3\mathbb{Z}$ in the ring $R = \mathbb{Z}$. Then $R \bowtie I \subset R \bowtie J$ is a minimal integral extension of the decomposed case, with (for instance) $q = (0, 3)$.

Proposition 3.4.6. *Let $R \bowtie I \subset R \bowtie J$ be a minimal extension of bowtie rings (which is necessarily integral by the forthcoming Corollary 6.1.3). Following the terminology of Theorem 3.4.3, this is an extension in the ramified case if and only if there exist $r \in R, j \in J \setminus I, M \in Max(R)$ such that $r^2 \in M$ and $2rj + j^2 \in I$. Otherwise, it is an extension in the decomposed case.*

Proof. Suppose that such an r, j, and M exist, and consider the element $q = (r, r + j) \in R \bowtie J \setminus R \bowtie I$. Then $R \bowtie J = (R \bowtie I)[q]$ by minimality. Further, $q^2 = (r^2, r^2 + 2rj + j^2)$ is contained in the maximal ideal $M' := \{(m, m + i) | m \in M, i \in I\}$ of $R \bowtie I$ (cf. [DFF, Proposition 2.6 (4)]). Note that $M'q = \{(rm, rm + mj + ri + ij) | m \in M, i \in I\}$ is contained in M' (since M and I are ideals of R). Then by Theorem 3.4.3 it follows that this is an extension in the ramified case.

Conversely, if $R \bowtie I \subset R \bowtie J$ is a ramified extension, then by Theorem 3.4.3 and [DFF, Proposition 2.6 (4)] there exists an $M \in Max(R)$ and some $(r, r + j) \in R \bowtie J \setminus R \bowtie I$ such that $R \bowtie J = (R \bowtie I)[(r, r + j)]$ and $(r, r + j)^2 \in M' := \{(m, m + i) | m \in M, i \in I\}$. Clearly we must have that $j \in J \setminus I$ and since $(r, r + j)^2 = (r^2, r^2 + 2rj + j^2) \in M'$, we have that $r^2 \in M$ and $2rj + j^2 \in I$ by definition of M'.

Finally, if the extension $R \bowtie I \subset R \bowtie J$ does not satisfy the above conditions (or equivalently, is not an extension in the ramified case), then by Theorem 3.4.3 and Remark 3.4.5, it must be an extension in the decomposed case.

$\square$

3.5 A Note on Nagata's Idealization

Let R be a ring, and E be an $R-$module. As we have seen, Nagata's idealization is the ring $R \ltimes E := \{(r, e) | r \in R, e \in E\}$ where addition is defined as in the abelian group $R \oplus E$ and multiplication is defined as $(r, e)(s, f) = (rs, rf + se)$. If we define a trivial multiplication on E (so $ee' = 0$ for all $e, e' \in E$), then the set $R + E$ is a ring, and R embeds into this ring in the obvious fashion. Let us denote this embedding by ι. With the trivial multipication we see that E becomes an $R-$algebra (so an $\iota(R)-$algebra). Further, it is easy to see that the map $R \ltimes E \to R \bowtie^{\iota} E$ given by $(r, e) \mapsto (r, \iota(r) + e)$ is an isomorphism. Thus we may view $R \ltimes E$ as a general bowtie ring, $R \bowtie^{\iota} E$, so that Nagata's idealization becomes a special case of many of the results of this chapter. We will record some of the more notable results here.

The first order of business is to describe Lemma 3.2.1 and Theorem 3.2.3 in terms of idealizations, a result which in turn extends [D2, Remark 2.9]. We note that the basic proof concept in this cited remark is utilized in the proof of Lemma 3.2.1 from which much of this chapter follows. The next theorem is easily proven by the same methods as for the general bowtie ring case.

Theorem 3.5.1. *Let R be a ring and let $E \subseteq F$ be $R-$modules. Then every element of $[R \ltimes E, R \ltimes F]$ has the form $R \ltimes E'$ for some $R-$module E' with $E \subseteq E' \subseteq F$. Further, there is an order-isomorphism between the sets $[R \ltimes E, R \ltimes F]$ and $[E, F]$, given by the map $R \ltimes E' \mapsto E'$.*

Corollary 3.5.2. *Let R be a ring and let $N \subseteq M$ be $R-$modules. Then $R \ltimes N \subset R \ltimes M$ is a minimal ring extension if and only if M/N is a simple $R-$module.*

Corollary 3.5.3. *Let R be a ring and let $N \subseteq M$ be $R-$modules. Then $R \ltimes N \subset R \ltimes M$ is a $\Delta-$extension.*

Numerous related properties on idealizations were recently presented by Gabriel Picavet and Martine Picavet-L'Hermitte. We list some of these results in the next corollary, noting that these follow as special cases of applications of Theorem 3.2.3 (namely, the results recorded in Corollary 3.2.4). Given a ring R and an $R-$module M, we will use $L_R(M)$ to denote the length of M (i.e., the number of strict inclusions in the longest chain of submodules of M). For the following corollary, note that $R \cong R \ltimes 0$. In the first assertion of the corollary, note that every $R-$module lying between 0 and R (as sets) is an ideal of R.

Corollary 3.5.4. *Let R be a ring and $N \subset M$ be $R-$modules. Then*

- *$R \subset R \ltimes R$ has FCP if and only if R is Artinian.*
- *$R \subset R \ltimes M$ has FIP if and only if M has only finitely many submodules.*
- *$R \subset R \ltimes M$ has FCP if and only if $L_R(M) < \infty$.*
- *$R \ltimes N \subset R \ltimes M$ has FIP if and only if M/N has only finitely many submodules.*
- *$R \ltimes N \subset R \ltimes M$ has FCP if and only if $L_R(M/N) < \infty$.*

Proof. The first three statements are immediate. For the rest, note that the set of $R-$submodules of M/N is the set of $R-$modules of the form N'/N for an intermediate $R-$module $N \subseteq N' \subseteq M$.

□

Finally, we consider the analogous results to Lemma 3.2.13 and Theorem 3.2.14 in the context of Nagata's idealization. The proof uses the same process so we omit it for brevity.

Theorem 3.5.5. *Let $R \subset T$ be a ring extension and let M be a $T-$module. Then every ring in $[R \ltimes M, T \ltimes M]$ has the form $S \ltimes M$ for some ring $S \in [R, T]$. Further, there is an order-isomorphism from $[R \ltimes M, T \ltimes M]$ to $[R, T]$, given by the map $S \ltimes M \mapsto S$.*

Again we provide a list of relevant consequences in the specific case of idealizations. These also all have analogues in Section 3.2 in the context of general bowtie rings.

Corollary 3.5.6. *Let $R \subset T$ be rings and M a $T-$module. Then*

- $R \ltimes M \subset T \ltimes M$ *is a minimal extension if and only if* $R \subset T$ *is a minimal extension.*

- $R \ltimes M \subset T \ltimes M$ *has FCP if and only if* $R \subset T$ *has FCP.*

- $R \ltimes M \subset T \ltimes M$ *has FIP if and only if* $R \subset T$ *has FIP.*

- $R \ltimes M \subset T \ltimes M$ *is a* $\Delta-$*extension if and only if* $R \subset T$ *is a* $\Delta-$*extension.*

- $R \ltimes M \subset T \ltimes M$ *is a* $\lambda-$*extension if and only if* $R \subset T$ *is a* $\lambda-$*extension.*

Chapter 4: Flat Epimorphisms

4.1 Integrally Closed Minimal Extensions

Let R be a ring. Given R-modules L, M, N a *short exact sequence* is a chain of R-module homomorphisms

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

such that α is injective and β is surjective. We say that an $R-$module A is *flat* if for any short exact sequence as above, the induced sequence obtained by tensoring each module with A, namely

$$0 \longrightarrow A \otimes_R L \xrightarrow{\iota \otimes \alpha} A \otimes_R M \xrightarrow{\iota \otimes \beta} A \otimes_R N \longrightarrow 0,$$

is also a short exact sequence. (We present this only as the classical definition of flatness; with the aid of known equivalent properties, we will in fact never need to use the definition in any of the proofs in this chapter.)

Let $f : A \to B$ be a ring homomorphism. Then we say that f is an *epimorphism* if for any ring homomorphisms $g, h : B \to C$, $g \circ f = h \circ f$ implies $g = h$. In this sense, epimorphisms generalize surjective homomorphisms; the canonical map $R \to R/I$ (for a ring R with ideal I) is commonly given as an example of an epimorphism. If $A \subseteq B$ is a ring extension and the inclusion map $f : A \to B$ is an epimorphism, we will say that $A \subseteq B$ is an *epimorphic extension* (or simply an *epimorphism*, when the context is obvious).

By saying that a ring homomorphism $f : A \to B$ is a *flat epimorphism* (in particular, when B is a ring extension of A), we mean that f is an epimorphism making B a flat $A-$module (via the action $a \cdot b := f(a)b$ for each $a \in A, b \in B$). If S is a multiplicatively closed set in R, then the natural map $R \to R_S$ is an example of a flat epimorphism. Thus if S contains only regular elements, then $R \subset R_S$ is a ring extension that is a flat epimorphism (as expected, we may refer to this as a *flat epimorphic extension*).

Let $R \subset T$ be a minimal extension, so $T = R[u]$ for each $u \in T \setminus R$. Clearly this extension is integral or R is integrally closed in T. If R is integrally closed in T, then this extension is a flat epimorphism by the following lemma, which can be derived by combining [K, Theorem 44] with [FO, Theorem 2.2 ((ii)(e) and (iii))].

Lemma 4.1.1. *Let $R \subset T$ be a minimal extension. Then $R \subset T$ is either an integral extension or a flat epimorphism. Thus every minimal integrally closed extension of rings is a flat epimorphism.*

We will use this fact to determine when certain minimal extensions are integral, by showing that they are not flat epimorphisms. The next lemma (a generalization of Lemma 1.5.8 that will only be necessary for the current section) is a consequence of [DFF, Proposition 2.6].

Lemma 4.1.2. *Let $f : R \to R'$ be a ring homomorphism with I and ideal of R'. Then the prime ideals of $R \bowtie^f I$ are exactly the sets of the following two forms:*

1. $\{(p, f(p) + i) | p \in P, i \in I\}$ *and*

2. $\{(a, f(a) + i) | a \in R, i \in I, f(a) + i \in Q\}$

where the P run through all primes of R and the Q run through all primes of R'.

Lemma 4.1.3. *Let $f : R \to R'$ be a ring homomorphism with $I \subset J$ ideals of R'. Let $P \in Spec(R)$ and $Q \in Spec(R')$. If $Q_0 = \{(p, f(p) + j) | p \in P, j \in J\}$ and $Q_1 = \{(a, f(a) + j) | a \in R, j \in J, f(a) + j \in Q\}$ in $Spec(R \bowtie^f J)$, then*

$$P_0 := Q_0 \cap (R \bowtie^f I) = \{(p, f(p) + i) | p \in P, i \in I\}$$

and

$$P_1 := Q_1 \cap (R \bowtie^f I) = \{(a, f(a)+i) | a \in R, i \in I, f(a)+i \in Q\}.$$

Proof. Since $I \subset J$ it is clear that $P_0 \subseteq Q_0$. Obviously P_0 is contained in $R \bowtie^f I$ and thus in the intersection $Q_0 \cap (R \bowtie^f I)$. Conversely, let $(p, f(p)+j)$ lie in Q_0. If it also lies in $R \bowtie^f I$, then we must have $j \in I$. It follows that $Q_0 \cap (R \bowtie^f I) \subseteq P_0$ and so these primes are equal.

Next, since $I \subset J$ it is clear that $P_1 \subseteq Q_1$. Further, since $P_1 \subseteq R \bowtie^f I$ we have that $P_1 \subseteq Q_1 \cap (R \bowtie^f I)$. Now let $(a, f(a)+j) \in Q_1$. If it is also in $R \bowtie^f I$, then we must have $j \in I$. Also $f(a)+j \in Q$ by definition of Q_1. Thus $Q_1 \cap (R \bowtie^f I) \subseteq \{(a, f(a)+i) | a \in R, i \in I, f(a)+i \in Q\} = P_1$.

□

Lemma 4.1.4. *Let $f : R \to R'$ be a ring homomorphism with $I \subset J$ ideals of R'. Suppose for some $Q \in Spec(R')$ that $I \subseteq Q, J \nsubseteq Q$. Then the contraction mapping $Spec(R \bowtie^f J) \to Spec(R \bowtie^f I)$ is not one-to-one.*

Proof. Take such a Q and consider the primes

$$Q_0 = \{(a, f(a)+j) | a \in R, j \in J, f(a)+j \in Q\}$$

and

$$Q_1 = \left\{(p, f(p)+j) | p \in f^{-1}(Q), j \in J\right\}$$

in $Spec(R \bowtie^f J)$. These are distinct, since by assumption there exists $j \in J \setminus Q$, and thus $(0, j) \in Q_1 \setminus Q_0$. But we claim these primes have the same contraction in $R \bowtie^f I$.

By the Lemma 4.1.3, $P_0 := Q_0 \cap (R \bowtie^f I) = \{(a, f(a)+i) | a \in R, i \in I, f(a)+i \in Q\}$, and $P_1 := Q_1 \cap (R \bowtie^f I) = \left\{(p, f(p)+i) | p \in f^{-1}(Q), i \in I\right\}$. Since $f(f^{-1}(Q)) \subseteq Q$ and $I \subseteq Q$, we have for each $(p, f(p)+i) \in P_1$ that $f(p)+i \in Q$, i.e., $P_1 \subseteq P_0$. As for the reverse inclusion, note for each $(a, f(a)+i) \in P_0$, that $i \in I \subseteq Q$, so $f(a) \in Q$, i.e. $a \in f^{-1}(Q)$. It follows that $P_0 \subseteq P_1$ and we have equality.

$\square$

Proposition 4.1.5. *Let $f : R \to R'$ be a ring homomorphism with $I \subset J$ ideals of R'. Suppose that $R \bowtie^f I \subset R \bowtie^f J$ be a minimal extension. If there exists a $Q \in Spec(R')$ such that $I \subseteq Q$ but $J \nsubseteq Q$, then the extension $R \bowtie^f I \subset R \bowtie^f J$ is integral.*

Proof. Suppose that such a Q exists. Then by Lemma 4.1.4 the contraction mapping $Spec(R \bowtie^f J) \to Spec(R \bowtie^f I)$ is not one-to-one. Thus by [L, Chapitre IV, Proposition 1.4] the embedding $R \bowtie^f I \to R \bowtie^f J$ is not an epimorphism. Since every minimal extension is either integral or a flat epimorphism it follows that $R \bowtie^f J$ is integral over $R \bowtie^f I$.

$\square$

Corollary 4.1.6. *Let $f : R \to R'$ be a ring homomorphism with I and ideal of R'. If I is prime in R', then every minimal extension of the form $R \bowtie^f I \subset R \bowtie^f J$ (with J an ideal of R' containing I) is integral.*

4.2 General Epimorphisms and Flatness

As we have already noted in the previous section, every minimal extension is either integral or a flat epimorphism. Suppose that $R \subset T$ is a ring extension and that J is a common ideal to R and T. Then $R \subset T$ is a non-integral minimal extension if and only if $R \bowtie J \subset T \bowtie J$ is a non-integral minimal extension (Corollary 3.2.15 and the forthcoming Corollary 6.1.2), and either hypothesis implies that $R \subset T$ and $R \bowtie J \subset T \bowtie J$ are flat epimorphisms (Lemma 4.1.1). We now want to investigate flat epimorphisms in a more general setting, without making assumptions on integrality or minimality. Further, for the sake of generality most of the results of this section will be proved for epimorphic (resp., flat epimorphic) maps which are not necessarily embeddings (i.e., we temporarily stray away from the study of ring extensions to the more general context of ring homomorphisms). However, as some of the tools used have no analogues for general bowtie rings, we will adhere to studying simple bowtie rings in this section.

We begin the section simply with epimorphisms. After a few notes on flatness, we end with the main goal of this chapter, investigating flat epimorphisms.

Theorem 4.2.1. *Let $f : R \to T$ be a ring homomorphism, and I, J ideals of R, T, respectively, such that $f(I) \subseteq J$. Define $f' : R \bowtie I \to T \bowtie J$ as $f'(r, r+i) = (f(r), f(r) + f(i))$. If f' is an epimorphism then so f. The converse holds if and only if $J = f(I)T$.*

Proof. Suppose f' is an epimorphism. Let $g, h : T \to C$ be ring homomorphisms satisfying that $g \circ f = h \circ f$. We can extend g, h to homomorphisms $g', h' : T \bowtie J \to C$ by setting $g'(t, t+j) = g(t), h'(t, t+j) = h(t)$. By construction $g' \circ f' = h' \circ f'$ and since f' is an epimorphism we have that $g' = h'$. Then by definition of these maps we see that for all $t \in T$, $g(t) = h(t)$. Thus the map f is an epimorphism.

To analyze the converse, suppose that f is an epimorphism, and note that $f(I)T \subseteq JT = J$.

First assume the strict inclusion $f(I)T \subset J$; we will show that f' is not an epimorphism in this case. Define the maps $g, h : T \bowtie J \to T/f(I)T$ by $g(t, t+j) = \overline{t}$, $h(t, t+j) = \overline{t+j}$ (where for any element $s \in T$, $\overline{s}$ denotes the canonical image of s in $T/f(I)T$). It is clear that these are both homomorphisms, as each is simply a composition of the projection map onto one coordinate with the canonical map from T onto $T/f(I)T$. Note that $g \neq h$, since for any $j \in J \setminus f(I)T$ we have $g(0, j) = \overline{0}$ and $h(0, j) = \overline{j} \neq \overline{0}$. However for any element $(r, r+i) \in R \bowtie I$ we have that $g(f(r, r+i)) = g(f(r), f(r) + f(i)) = \overline{f(r)}$ and $h(f(r, r+i)) = h(f(r), f(r) + f(i)) = \overline{f(r) + f(i)} = \overline{f(r)}$, so that $g \circ f = h \circ f$. It follows that the map f is not an epimorphism.

Now instead, assume that $f(I)T = J$. Let $g', h' : T \bowtie J \to C$ be ring homomorphisms such that $g' \circ f' = h' \circ f'$. Then $g(f(r, r+i)) = h(f(r, r+i))$ for all $(r, r+i) \in R \bowtie I$. In particular $g(f(r, r)) = h(f(r, r))$ for all $r \in R$. Define the mapping $d : f(R) \to f(R)^{\Delta}$ as $d(f(r)) = (f(r), f(r))$. Clearly d is an isomorphism. Now for all $r \in R$,

$$(g' \circ f')(r,r) = (h' \circ f')(r,r)$$

$$g'(f(r), f(r)) = h'(f(r), f(r))$$

$$(g' \circ d)(f(r)) = (h' \circ d)(f(r)),$$

and since f is an epimorphism, $g' \circ d = h' \circ d$ on all of T; that is, $g' = h'$ on the diagonal image of T in $T \bowtie J$.

Now for any $(t, t+j) \in T \bowtie f(I)T$, note that we can write this element as $(t,t) + (0, \sum_{k=1}^{n} f(i_k)t_k) = (t,t) + \sum_{k=1}^{n}(0, f(i_k))(t_k, t_k) = (t,t) + \sum_{k=1}^{n} f'(0, i_k)(t_k, t_k)$ for some $i_k \in I, t_k \in T, n \in \mathbb{N}$. Then (letting $h'f' := h' \circ f'$ and $g'f' := g' \circ f'$) we have

$$g'(t, t+j) = g'(t,t) + g'(0,j) = h'(t,t) + g'(0,j)$$

$$= h'(t,t) + g'(\sum_{k=1}^{n} f'(0,i_k)(t_k,t_k)) = h'(t,t) + \sum_{k=1}^{n} g'f'(0,i_k)g'(t_k,t_k) =$$

$$h'(t,t) + \sum_{k=1}^{n} h'f'(0,i_k)h'(t_k,t_k) = h'(t,t) + h'(\sum_{k=1}^{n} f'(0,i_k)(t_k,t_k))$$

$$= h'(t,t) + h'(\sum_{k=1}^{n}(0, f(i_k))(t_k,t_k)) = h'((t,t) + \sum_{k=1}^{n}(0, f(i_k))(t_k,t_k)) = h'(t, t+j),$$

so $g' = h'$ on all of $T \bowtie f(I)T$. It follows that the map f' is an epimorphism.

□

Corollary 4.2.2. *Let $I \subset J$ be ideals of a ring R. Then the extension $R \bowtie I \subset R \bowtie J$ is not an epimorphism. In particular, for any nonzero proper ideal I of a ring R neither $R^{\Delta} \subset R \bowtie I$ nor $R \bowtie I \subset R \times R$ $(= R \bowtie R)$ is an epimorphism.*

Corollary 4.2.3. *Let $R \subset T$ be an extension of rings with I and ideal of R. If $R \bowtie I \subset T \times T$ is an epimorphism then I contains a finitely generated dense ideal of R.*

Proof. Recall that $T \times T = T \bowtie T$. Now supposing $R \bowtie I \subset T \bowtie T$ is an epimorphism, we must have that $IT = T$, by Theorem 4.2.1. The statement then follows from [DS6, Lemma 2.4]. $\square$

As we have noted, our present goal is to investigate flat epimorphisms, but first we will record a few quick results on flatness alone.

Proposition 4.2.4. *Let $R \subseteq T$ be a ring extension with J an ideal of T. Suppose R is a PID and that every nonzero element of R is a regular element in T. Then $R^{\Delta} \subseteq T \bowtie J$ is a flat extension.*

Proof. Suppose $(r, r)(t, t + j) = (0, 0)$ for some $0 \neq r \in R$, $(t, t + j) \in T \bowtie J$. Then $rt = 0$, and so by assumption $t = 0$. Now we have $(r, r)(0, j) = (0, 0)$ implying that $rj = 0$. Again by assumption $j = 0$, so that $(t, t + j) = (0, 0)$. It follows that $T \bowtie J$ is torsion-free as an $R^{\Delta}-$module, so by [R, Corollary 3.50], $T \bowtie J$ is flat over R^{Δ}.

$\square$

Corollary 4.2.5. *If R is a PID then for every ideal I of R, the extension $R^{\Delta} \subset R \bowtie I$ is flat.*

Lemma 4.2.6. *Let $f : R \to T$ be a ring homomorphism, and I, J ideals of R, T, respectively, such that $f(I) \subseteq J$. Define $f' : R \bowtie I \to T \bowtie J$ as $f'((r, r+i)) = (f(r), f(r)+f(i))$. If f' is a flat ring map then so is f.*

Proof. Suppose that the map f' is flat, and suppose that we have the sum $\sum_{k=1}^{n} t_k f(r_k) = 0$ where $t_k \in T$, $r_k \in R$. Thus $\sum_{k=1}^{n} (t_k, t_k)(f(r_k), f(r_k)) = \sum_{k=1}^{n} (t_k, t_k) f'(r_k, r_k) = (0, 0)$, with $(t_k, t_k) \in T \bowtie J, (r_k, r_k) \in R \bowtie I$. Now by [R, Lemma 3.65(i) $\Rightarrow$ (iii)], there exists $(t'_q, t'_q + j'_q) \in T \bowtie J$ for $q = 1, ..., m$ (some fixed integer m), and $(s_{qk}, s_{qk} + e_{qk}) \in R \bowtie I$ with $\sum_k f'(s_{qk}, s_{qk} + e_{qk}) f'(r_k, r_k) = \sum_k (f(s_{qk}), f(s_{qk}) + f(e_{qk}))(f(r_k), f(r_k)) = (0, 0)$ for all q and $\sum_{q=1}^{m} (t'_q, t'_q + j'_q) f'(s_{qk}, s_{qk} + e_{qk}) = \sum_{q=1}^{m} (t'_q, t'_q + j'_q)(f(s_{qk}), f(s_{qk}) + f(e_{qk})) = (t_k, t_k)$ for all k.

Then looking only at the first coordinates it follows that $\sum_k f(s_{qk})f(r_k) = 0$ for all q and $\sum_{q=1}^{m} t'_q f(s_{qk}) = t_k$ for all k. Now applying [R, Lemma 3.65(iii) $\Rightarrow$ (i)], we have that T is flat as an $f(R)-$module, so f is a flat map.

$\square$

The flatness of an extension of bowtie rings $R \bowtie I \subset T \bowtie J$ does not follow easily from the flatness of $R \subseteq T$. By [M2, p. 33], any flat ring extension satisfies going-down, so even for $R = T$ (which is trivially flat over itself), we will see (in Theorem 6.3.7) that the extension $R \bowtie I \subset R \bowtie J$ must satisfy rather complicated assumptions on the ideals I and J to imply that the ring extension satisfies going-down; thus complicated assumptions are necessary for the extension to even possibly be flat. In particular, setting $R = T$, then Example 6.3.10 gives an extension where T is flat over R (trivially) but $T \bowtie J$ is not flat over $R \bowtie I$.

If we restrict ourselves to flat epimorphisms, however, then we do find a strong correlation between the extensions $R \subset T$ and $R \bowtie I \subset T \bowtie J$, or more generally between any given homomorphism $R \to T$ and the map that it induces from $R \bowtie I$ to $T \bowtie J$. Before proving this correlation, we will need the following lemma.

Lemma 4.2.7. *Let $f : R \to T$ be a ring homomorphism, and let I be an ideal of R. Define $f' : R \bowtie I \to T \bowtie IT$ by $f'(r, r+i) = (f(r), f(r) + f(i))$. Let $P' \in Spec(R \bowtie I)$, where P' is constructed from $P \in Spec(R)$ in the sense of Lemma 1.5.8. If $f(P)T = T$, then $f'(P')(T \bowtie IT) = T \bowtie IT$.*

Proof. Clearly $f'(P')(T \bowtie f(I)T) \subseteq T \bowtie f(I)T$. We wish the show that the reverse inclusion holds as we assume that $f(P)T = T$.

We know that P' must be of one of the two forms described in Lemma 1.5.8. In either case, fix the $P \in Spec(R)$ used to define P' (again in the sense of Lemma 1.5.8), and note that P' contains the diagonal image $\{(p,p)|p \in P\}$ of P in $R \bowtie I$. Let x be an arbitrary element of $T \bowtie f(I)T$. Then x has the form $(t, t + \sum f(i_k)t_k)$ for $t, t_k \in T, i_k \in I$. Rewrite this as $(t,t) + \sum(0, f(i_k)t_k)$. Since $f(P)T = T$ we see that $(t,t) = (\sum f(p_j)t_j, \sum f(p_j)t_j) =$

$\sum(f(p_j), f(p_j))(t_j, t_j) = \sum f'(p_j, p_j)(t_j, t_j) \in f'(P')(T \bowtie f(I)T)$. Thus we only need to show that $\sum(0, f(i_k)t_k) \in f'(P')(T \bowtie f(I)T)$. We will prove something slightly stronger and show that $(0, f(i_k)t_k) \in f'(P')(T \bowtie f(I)T)$ for every k; thus we will abuse notation and drop the subscript k for clarity. Now since $f(P)T = T$, we can write (for some $p_l \in P$, $t_l \in T$)

$$(0, it) = (0, f(i)\sum p_l t_l) = (0, \sum p_l f(i)t_l)) =$$

$$\sum(0, p_l(f(i)t_l)) = \sum(p_l, p_l)(0, f(i)t_l) \in P'(T \bowtie f(I)T).$$

Thus $T \bowtie f(I)T \subseteq P'(T \bowtie f(I)T)$ and we have that the sets are equal.

□

Lemma 4.2.8. *Let $f : R \to T$ be a ring homomorphism and let I, J be ideals of R, T, respectively such that $f(I) \subseteq J$. Define $f' : R \bowtie I \to T \bowtie f(I)T$ as $f'(r, r+i) = (f(r), f(r) + f(i))$. Define $h : T \bowtie J \to T$ as the projection to the first coordinate, i.e., $h(t, t+j) = t$. If $P' \in Spec(R \bowtie I)$ lies over $P \in Spec(R)$ (as in Lemma 1.5.8), then $(h \circ f')(R \bowtie I \setminus P') = f(R \setminus P)$.*

Proof. Let $y \in f(R \setminus P)$. Then there exists an $x \in R \setminus P$ with $f(x) = y$. Note that $(x, x) \in R \bowtie I \setminus P'$ by Lemma 1.5.8. Now $(y, y) = (f(x), f(x)) = f'(x, x) \in f'(R \bowtie I \setminus P')$, so $y = (h \circ f')(x, x) \in (h \circ f')(R \bowtie I \setminus P')$.

Conversely, let $y \in (h \circ f')(R \bowtie I \setminus P')$. Then there is an element $(x, x+e) \in R \bowtie I \setminus P'$ with $(h \circ f')(x, x+e) = y$. But $(h \circ f')(x, x+e) = h(f(x), f(x) + f(e)) = f(x)$, so that $y = f(x)$. Clearly $x \in R$, but it is not possible that $x \in P$ or else we would have $(x, x+e) \in P'$. It follows that $y = f(x) \in f(R \setminus P)$.

□

The proof of the following proposition essentially mimics the proofs found in [D, Proposition 2.7(a)] and [DS3, Proposition 4.4(b)]. We generalize to the context of ring homomorphisms as studied in the current section, and for the sake of clarity, provide details that were taken as obvious (or unnecessary) in the original proofs. As usual, given a ring homomorphism $f : R \to T$ and a prime $P \in Spec(R)$ and we use $f_P : R_P \to T_{f(R \setminus P)}$ to denote the ring homomorphism $f_P(\frac{r}{s}) = \frac{f(r)}{f(s)}$.

Proposition 4.2.9. *Let $f : R \to T$ be a ring homomorphism and let I, J be ideals of R, T, respectively such that $f(I) \subseteq J$. Define $f' : R \bowtie I \to T \bowtie J$ as $f'((r, r+i)) = (f(r), f(r)+f(i))$. Let $P \in Spec(R)$ and let P' be a prime of $R \bowtie I$ lying over $P \in Spec(R)$ as in Lemma 1.5.8. Then:*

1. *If $I \subseteq P$, then $(R \bowtie I)_{P'} \cong R_P \bowtie I_P$ and $(T \bowtie J)_{f'((R \bowtie I)\setminus P')} \cong T_{f(R\setminus P)} \bowtie J_{f(R \setminus P)}$.*

2. *If $I \nsubseteq P$, then $(R \bowtie I)_{P'} \cong R_P$ and $(T \bowtie J)_{f'(R \bowtie I \setminus P')} \cong T_{f(R \setminus P)}$.*

Proof. (1.) Suppose that $I \subseteq P$. The conclusion that $(R \bowtie I)_{P'} \cong R_P \bowtie I_P$ is proved directly in [D, Proposition 2.7(a.)]. We will now show that $(T \bowtie J)_{f'((R \bowtie I)\setminus P')} \cong T_{f(R\setminus P)} \bowtie J_{f(R\setminus P)}$. Let $\phi : T \bowtie J \to T_{f(R \setminus P)} \bowtie J_{f(R\setminus P)}$ be the homomorphism defined by $(t, t+j) \mapsto (\frac{t}{1}, \frac{t}{1} + \frac{j}{1})$. We claim that every element of $(\phi \circ f')(R \bowtie I \setminus P')$ is a unit in $T_{f(R\setminus P)} \bowtie J_{f(R\setminus P)}$. Let $(r, r+i) \in R \bowtie I \setminus P'$. Then we must have that $r \in R \setminus P$ and so $(\frac{r}{1}, \frac{r}{1} + \frac{i}{1})$ is a unit in $R_{R\setminus P} \bowtie IR_{R \setminus P}$ (as in the proof of [D, Proposition 2.7(a.)]). Thus $(\phi \circ f')(r, r+i) = f'_{P'}(\frac{r}{1}, \frac{r}{1} + \frac{i}{1})$ is a unit in $f(R)_{f(R\setminus P)} \bowtie f(I)f(R)_{f(R\setminus P)} \subseteq T_{f(R\setminus P)} \bowtie f(I)T_{f(R\setminus P)}$.

Hence, by the Universal Mapping Property of ring localizations, there exists a unique extension $\Psi : (T \bowtie f(I)T)_{f'(R \bowtie I \setminus P')} \to T_{f(R\setminus P)} \bowtie f(I)T_{f(R \setminus P)}$ of ϕ given by $\Psi(\frac{a}{b}) = \phi(a)\phi(b)^{-1}$. We only need to show that this is an isomorphism. To show that it is onto, let $(\frac{t}{s}, \frac{t}{s} + \frac{j}{z}) \in T_{f(R\setminus P)} \bowtie f(I)T_{f(R\setminus P)}$. Since s, z are both in the multiplicatively closed set $f(R \setminus P)$ of T, so is their product sz; in particular, $sz \neq 0$. Now we can take the element $\frac{(tz, tz+js)}{(sz, sz)}$ to map to $(\frac{t}{s}, \frac{t}{s} + \frac{j}{z})$ via Ψ. To show that the map is one-to-one, suppose that

$\Psi(t, t+j) = (0,0)$. Then by definition of localization it follows that there exist elements u, v in the multiplicatively closed set $f(R \setminus P)$ satisfying $ut = 0$ and $v(t+j) = 0$; hence $(uv, uv)(t, t+j) = (0,0)$, so (by the definition of localization and the observation that (uv, uv) lies in the multiplicatively closed set $f'(R \bowtie I \setminus P))$ we have that $(t, t+j) = (0,0)$ in $(T \bowtie f(I)T)_{f'(R \bowtie I \setminus P')}$. Thus Ψ has zero kernel, so it is an injection.

(2.) Now suppose $I \nsubseteq P$. The proof that $(R \bowtie I)_{P'} \cong R_P$ is shown in [DS3, Proposition 4.4(b.)], so we proceed to show that $(T \bowtie J)_{f'(R \bowtie I \setminus P')} \cong T_{f(R \setminus P)}$. By Lemma 1.5.8 there are two cases to consider. For the first case, we assume that P' has the form $P' = \{(p, p+i) | p \in P, i \in I\}$. The surjective ring homomorphism $h : T \bowtie J \to T$ defined by $(t, t+j) \mapsto t$ has kernel equal to $0 \oplus J$. If $j \in I \setminus P$, then $(j, 0) \in R \bowtie I \setminus P'$, so that $(f(j), 0) = f'(j, 0) \in f'(R \bowtie I \setminus P')$. Since $(f(j), 0)(0 \oplus J) = (0,0)$ we conclude that $0 \oplus J$ is in contained in the kernel of the canonical map $T \bowtie J \to (T \bowtie J)_{f'(R \bowtie I \setminus P')}$. Now Lemma 4.2.8 gives that $h(f'(R \bowtie I \setminus P')) = f(R \setminus P)$, and we conclude by [DS3, Lemma 4.3] that $(T \bowtie J)_{f'(R \bowtie I \setminus P')} \cong T_{f(R \setminus P)}$.

Now for the second case, we assume that P' has the form $P' = \{(p+i, p) | p \in P, i \in I\}$. Consider the surjective map $h : T \bowtie J \to T$ given by $(t, t+j) \mapsto t+j$. The kernel of this map is $J \oplus 0$. Given $j \in I \setminus P$, then $(0, f(i)) = f'(0, i) \in f'(R \bowtie I \setminus P')$. Since $(0, f(i))(J \oplus 0) = (0,0)$ we see that $J \oplus 0$ is contained in the kernel of the canonical map $T \bowtie J \to (T \bowtie J)_{f'(R \bowtie I \setminus P')}$. As in the preceding paragraph, $h(f'(R \bowtie I \setminus P')) = f(R \setminus P)$, and so by [DS3, Lemma 4.3] we again have that $(T \bowtie J)_{f'(R \bowtie I \setminus P')} \cong T_{f(R \setminus P)}$.

□

Theorem 4.2.10. *Let $f : R \to T$ be a ring homomorphism and let I, J be ideals of R, T, respectively such that $f(I) \subseteq J$. Define $f' : R \bowtie I \to T \bowtie f(I)T$ as $f'(r, r+i) = (f(r), f(r) + f(i))$. If f' is a flat epimorphism then so is f. The converse holds if and only if $J = f(I)T$.*

Proof. First suppose that f' is a flat epimorphism. Then by Theorem 4.2.1 and Lemma

4.2.6, f is a flat epimorphism as well.

Now conversely, suppose that f is a flat epimorphism. By Theorem 4.2.1, for the map f' to be a flat epimorphism the condition $J = f(I)T$ is necessary (otherwise the extension is not even an epimorphism). We will show that it is also sufficient; that is, we will show that the mapping $f' : R \bowtie I \to T \bowtie f(I)T$ is indeed a flat epimorphism.

By [L, Proposition 2.4] (or [G3, Theorem 1.2.21]) f being a flat epimorphism is equivalent to the following condition: that for every $P \in Spec(R)$, either $f(P)T = T$ or $R_P \cong T_{f(R\setminus P)}$ (via the map f_P). We want to utilize this same theorem by showing that for any given $P' \in Spec(R \bowtie I)$, we have either $f'(P')(T \bowtie f(I)T) = T \bowtie f(I)T$ or $(R \bowtie I)_{P'} \cong (T \bowtie IT)_{f'(R\bowtie I\setminus P')}$ (via the map ${f'}_{P'}$). Thus let $P' \in Spec(R \bowtie I)$ be constructed from $P \in Spec(R)$ in the sense of Lemma 1.5.8.

If $f(P)T = T$, then we have by Lemma 4.2.7 that $f'(P')(T \bowtie f(I)T) = T \bowtie f(I)T$. Otherwise, $R_P \cong T_{f(R\setminus P)}$. Then we have two cases to consider, $I \subseteq P$ or $I \not\subseteq P$. If $I \subseteq P$, then by [Proposition 4.2.9 (a.)], we have

$$(R \bowtie I)_{P'} \cong R_P \bowtie I_P = R_P \bowtie IR_P \cong T_{f(R\setminus P)} \bowtie f(I)T_{f(R\setminus P)} \cong (T \bowtie f(I)T)_{f'(R\bowtie I\setminus P')}.$$

If instead we have that $I \not\subseteq P$, then by [Proposition 4.2.9 (b.)], $(R \bowtie I)_{P'} \cong R_P \cong T_{f(R\setminus P)} \cong (T \bowtie f(I)T)_{f'(R\bowtie I\setminus P')}$. In any case, the assumption that $R_P \cong T_{f(R\setminus P)}$ for all $P \in Spec(R)$ implies that $(R \bowtie I)_{P'} \cong (T \bowtie f(I)T)_{f'(R\bowtie I\setminus P')}$ for all $P' \in Spec(R \bowtie I)$. It now follows from [L, Proposition 2.4] (or, again, [G1, Theorem 1.2.21]) that f' is a flat epimorphism.

$\square$

Let I be an ideal of a ring R, and let S be a multiplicatively closed set in R. It is not clear if $R_S \bowtie I_S$ will always be a localization of $R \bowtie I$ at some multiplicatively closed set in $R \bowtie I$, but in any case we now have the following.

Corollary 4.2.11. *Let I be an ideal of a ring R and let S be a multiplicatively closed set in R. Then the natural map $R \bowtie I \to R_S \bowtie I_S$ given by $(r, r+i) \mapsto (\frac{r}{1}, \frac{r}{1} + \frac{i}{1})$ is a flat* *epimorphism.*

Since the purpose of this document is to investigate ring extensions, for convenience we record the results of Theorems 4.2.1 and 4.2.10 in the context of ring extensions.

Theorem 4.2.12. *Let $R \subseteq T$ be a ring extension with ideals $I \subseteq J$, respectively. If $R \bowtie I \subseteq T \bowtie J$ is an epimorphism (resp., flat epimorphism) then $R \subseteq T$ is an epimorphism (resp. flat epimorphism). The converse (in either case) holds if and only if $J = IT$.*

4.3 Examples

We now return to the context of ring extensions. We will use our results from the previous section to construct new examples of flat (respectively, flat epimorphic) extensions.

First we provide a quick example of an epimorphic extension that is not flat. As we have seen, for an extension $R \bowtie I \subset T \bowtie J$ to be an epimorphism or a flat epimorphism, it is necessary that $J = IT$. However, for the extension simply to be flat, that assumption is not always necessary, as in the following example (where $I = 0$ so that $IT = 0$). Since $\mathbb{Z}$ is a PID, this example is an immediate application of Corollaries 4.2.5 and 4.2.2.

Example 4.3.1. The extension $\mathbb{Z}^{\Delta} = \mathbb{Z} \bowtie 0 \subset \mathbb{Z} \bowtie 2\mathbb{Z}$ is flat, but not an epimorphism.

Remark 4.3.2. We can ostensibly use Theorem 4.2.12 to construct examples of non-minimal extensions that are flat epimorphisms from flat epimorphic minimal extensions. For instance, let $R \subset T$ be an integrally closed minimal extension (thus a flat epimorphism by Lemma 4.1.1), where R contains at least one ideal I that is not an ideal of T. Thus $I \subset IT$, so by Proposition 3.4.2, $R \bowtie I \subset T \bowtie IT$ is not a minimal extension, though by Theorem 4.2.10 it is a flat epimorphism.

The author believes that credit for the original extension in the following example belongs to J.R. Isbell (due to a public comment by Anton Geraschenko on an online message board), though this has not been verified.

Example 4.3.3. Let k be a field. Set $R = k[X, XY, XY^2 - Y]$ and $T = k[X, Y]$. It is known that $R \subset T$ is an epimorphic extension (where R and T are distinct rings) that is not a localization. In fact, no nonunit of R becomes a unit in T. With some work, it can also be shown that $R \subset T$ is a flat extension. The rings involved are domains, but we can use them to create flat epimorphic extensions of rings with zero-divisors which are not localizations.

The easiest way to do this is to create an extension of cross products, $R \times R \subset T \times T$. But we can also use nontrivial bowtie rings to give further examples. For instance, set $I = (X, XY, XY^2 - Y)$ and note that $IT = (X, Y)$. Thus $k[X, XY, XY^2 - Y] \bowtie (X, XY, XY^2 - Y) \subset k[X, Y] \bowtie (X, Y)$ is a flat epimorphic extension by Theorem 4.2.12.

We wish to show that this example is not a localization. Suppose that the extension $R \bowtie I \subset T \bowtie IT$ satisfies that $T \bowtie IT = (R \bowtie I)_S$ for some multiplicatively closed set S in $R \bowtie I$. Then S must not contain any zero-divisors of $R \bowtie I$ (or else the natural map from $R \bowtie I$ into $T \bowtie IT$ would not be an embedding). In particular, every element of S must be nonzero in both coordinates (since for instance an element $(0, s)$ could be killed by $(i, 0) \in R \bowtie I$ for any nonzero $i \in I$). But R is a domain, so this implies that every element of S must project to a regular element of R in each coordinate.

Now pick $(s, s + j) \in S$ such that $(s, s + j)$ is not a unit in $R \bowtie I$. We claim that s or $s + j$ is not a unit in R. Suppose otherwise, say $st = 1, (s + j)k = 1$. Then $(tk, tk) \in R \bowtie I$ and $(s, s + j)(tk, tk) = (k, t)$ is an element of $R \bowtie I$, which implies that $k - t \in I$, so that $(t, k) \in R \bowtie I$. But then $(s, s + j)$ has an inverse in $R \bowtie I$, a contradiction. This ends the proof of the claim.

Now we know that either s or $s+j$ is not a unit in R. Suppose s is not a unit. If $(s, s+j)$ is a unit in $T \bowtie IT$, then there are elements $t \in T, h \in IT$ such that $(s, s+j)(t, t+j) = (1, 1)$, and so $st = 1$. But then some nonunit of R becomes a unit in T, a contradiction. If instead $s + j$ is not a unit, then we similarly find a $t \in T, h \in IT$ with $(s, s + j)(t, t + h) = (1, 1)$, giving $(s + j)(t + h) = (1, 1)$ in T, the same contradiction.

It follows that no nonunit elements of $R \bowtie I$ become units in the extension $T \bowtie IT$, so that this extension cannot be a localization.

The same reasoning can be used to show that $T \bowtie IT$ is a flat epimorphic extension of $R \bowtie I$ that is not a localization (and where $R \bowtie I \neq T \bowtie IT$ since $R \neq T$) for any ideal I of R, in particular (X^n) for any natural number n. Thus the current example provides an infinite collection of distinct flat epimorphic ring extensions that are not localizations.

Chapter 5: Complemented Rings and Related Topics

5.1 Equivalent Properties

We say that a ring R satisfies *Property A* if every finitely generated ideal $I \subseteq Z(R)$ has a non-zero annihilator (in the literature such a ring is sometimes called a *McCoy ring*). We say that R has the *annihilator condition*, or the (a.c.), if given any two elements $a, b \in R$, there is a $c \in R$ satisfying that $Ann_R((a,b)) = Ann_R(c)$. Although Property A and the (a.c.) are related, neither property implies the other in general. Recall that for a ring R to be *von Neumann regular* we mean that for any $x \in R$, there exists a $y \in R$ such that $x^2y = x$. We say that R is *complemented* if its total quotient ring $tq(R)$ is von Neumann regular. When we say $Min(R)$ is compact, we mean as a subspace of $Spec(R)$ under the Zariski topology. The motivation for studying these properties together comes from the following theorem.

Theorem 5.1.1. *[H, Theorem 4.5] Let R be a reduced ring. Then the following properties are equivalent:*

- *R is complemented.*
- *$tq(R)$ is complemented.*
- *R has Property A and $Min(R)$ is compact.*
- *R has the (a.c.) and $Min(R)$ is compact.*

In this chapter we will study each of these properties, and search for equivalences between the two given rings in the extension $R \subset R \bowtie I$ (resp., $R \subset R \bowtie^f I$), where as usual we associate R with R^{Δ} (resp., with $\Gamma(f)$). This will become easier in the next section, where we assume that I is a regular ideal. At the end of this chapter we will take a closer look at complemented rings. In doing so, it will be useful to investigate the total quotient ring of a bowtie ring, in particular, describing the form of $tq(R \bowtie I)$ as it relates to $tq(R)$.

We will devote the rest of the current section to studying descent properties in the extension $R \subset R \bowtie I$. We will first look at the minimal primes of a ring R (denoted by $Min(R)$), and consider when this set is compact. We view $Min(R)$ as a topological subspace of $Spec(R)$ under the usual Zariski topology. That is, the basic open sets of $Spec(R)$ are all sets of the form $D(J) := \{P \in Spec(R) | J \nsubseteq P\}$ where J is an ideal of R.

Proposition 5.1.2. *Let I be and ideal of a ring R. If $Min(R \bowtie I)$ is compact, then $Min(R)$ is compact.*

Proof. Suppose that $Min(R \bowtie I) = \{P_\alpha\}$ is compact. It follows from [D, Proposition 5] that $Min(R \bowtie I) = \{P'_\alpha\} \cup \{P''_\alpha\}$ where $P'_\alpha := \{(p, p+i) | p \in P_\alpha, i \in I\}$, and $P''_\alpha := \{(p+i, p) | p \in P_\alpha, i \in I\}$.

Let $\bigcup D(I_\beta)$ be an open cover for $Min(R)$. Note for each ideal I_β of R here, the set $I_\beta \bowtie I$ is an ideal of $R \bowtie I$. We first claim that $D(I_\beta \bowtie I)$ is a(n open) cover for $Min(R \bowtie I)$.

Fix an α and consider P'_α, P''_α. From our cover for $Min(R)$ we have that there exists a β with $P_\alpha \in D(I_\beta)$, so $I_\beta \nsubseteq P_\alpha$, i.e. there exists an $x \in I_\beta \setminus P_\alpha$. Then $(x, x) \in I_\beta \bowtie I$ but $(x, x) \notin P'_\alpha$, and $(x, x) \notin P''_\alpha$. Thus $I_\beta \bowtie I$ is not contained in P'_α or P''_α. Thus $P'_\alpha \in D(I_\beta \bowtie I)$ and $P''_\alpha \in D(I_\beta \bowtie I)$.

It follows that $\bigcup D(I_\beta \bowtie I)$ is an open cover for $Min(R \bowtie I)$. Take a finite subcover, say (without loss of generality) $D(I_1 \bowtie I) \cup \cdots \cup D(I_n \bowtie I)$. We claim that $D(I_1) \cup \cdots \cup D(I_n)$ covers $Min(R)$.

Let $P_\alpha \in Min(R)$. Then $P'_\alpha \in Min(R \bowtie I)$ so there must be a $k \in \{1, ..., n\}$ where $P'_\alpha \in D(I_k \bowtie I)$, i.e. $I_k \bowtie I \nsubseteq P'_\alpha$. Thus there exists an $(i_k, i_k + i) \in I_k \bowtie I$ that is not in P'_α. If $i_k \in P_\alpha$, then surely $(i_k, i_k + i) \in P'_\alpha$, so we must have $i_k \notin P_\alpha$, so $I_k \nsubseteq P_\alpha$, and

$P_\alpha \in D(I_k)$. Hence the finite collection of open sets $D(I_1), ..., D(I_n)$ covers $Min(R)$, and it follows that $Min(R)$ is compact.

□

Next we study the descent of Property A, and this will suffice to show the descent of the property of being complemented.

Proposition 5.1.3. *Let I be an ideal of a ring R. If $R \bowtie I$ has Property A, then so does R.*

Proof. First suppose that $R \bowtie I$ has Property A, and let $(r_1, ..., r_n)$ be a finitely generated ideal of R consisting of zero-divisors. Then it is easy to see that $((r_1, r_1), ..., (r_n, r_n))$ is a finitely generated ideal of $R \bowtie I$ consisting of zero-divisors in this ring. Since $R \bowtie I$ has Property A, there is some nonzero $(s, s + j)$ annihilating this ideal. Then we can take whichever of s or $s + j$ is nonzero to annihilate $(r_1, ..., r_n)$.

□

Corollary 5.1.4. *Let I be an ideal of a ring R. If $R \bowtie I$ is complemented, then R is complemented.*

Proof. If $R \bowtie I$ is complemented then by Theorem 5.1.1 it has Property A and $Min(R \bowtie I)$ is compact. But then, as we have seen in Propositions 5.1.3 and 5.1.2, R has Property A and $Min(R)$ is compact, so that (by Theorem 5.1.1) R is complemented.

□

In the case of general bowtie rings, R and R' having one of the above properties does not necessarily imply that $R \bowtie^f I$ will have it. We will see numerous counterexamples in the next section. Whether the Property A and the (a.c.) always ascend in the case of simple bowtie rings is still unknown. However, as we will see at the end of this chapter, the condition that $R \bowtie I$ be complemented is equivalent to the condition that R be complemented.

5.2 Results Where I is Regular

We will proceed to study the related properties described in the previous section, now specifically for the case that the ideal I used in the construction $R \bowtie I$ is a regular ideal of R' (i.e., I contains a non-zero-divisor of R') and $f^{-1}(I)$ is a regular ideal of R. In this context the properties behave very nicely, even though we are working in the general bowtie ring construction $R \bowtie^f I$.

Lemma 5.2.1. *Let $R_1, ..., R_n$ be rings. Then $R_1 \times \cdots \times R_n$ has Property A if and only if each of the R_k has Property A.*

Proof. Clearly it suffices to prove the lemma for two rings. Thus consider $R \times S$ for two rings R and S.

First suppose that $R \times S$ has Property A. Let I and J be finitely generated ideals contained in $Z(R)$ and $Z(S)$, respectively. Then $I \times S$ is a finitely generated ideal of $R \times S$ and consists of zero-divisors (any element $(i, s) \in I \times S$ is annihilated by $(z, 0)$ where $z \in Ann_R(i)$). Since $R \times S$ has Property A, there exists some nonzero element $(x, y) \in R \times S$ annihilating $I \times S$. Clearly y must be zero. Thus x is nonzero, and we see that x annihilates I in R. The argument for J is identical and it follows that both R and S have Property A.

Now suppose that both R and S have Property A. Let H be a finitely generated ideal of $R \times S$ consisting of zero-divisors. Then $H = I \times J$ for some finitely generated (possibly improper) ideals I, J of R, S, respectively. If I consists of zero-divisors, then we can find an x in R annihilating I (since R has Property A). Then $(x, 0)$ annihilates H. If J consists of zero-divisors, then it is annihilated by some nonzero $y \in S$ and so the element $(0, y)$ annihilates H. Finally, if I and J are both regular ideals of R and S, respectively, then we can find a regular element $i \in I$ and a regular element $j \in J$. But then (i, j) is a regular element of H, a contradiction.

□

Corollary 5.2.2. *Let R be a zero-dimensional ring. Then $R \bowtie I$ has Property A for every (proper or improper) ideal I of R.*

Proof. Suppose $dim(R) = 0$ and that I is any proper ideal of I. Then by [D, Remark 1], $dim(R \bowtie I) = 0$. Thus by [H, Corollary 2.12], $R \bowtie I$ satisfies Property A. Finally, the case where $I = R$ follows from [H, Corollary 2.12] and Lemma 5.2.1 above, recalling that $R \bowtie R = R \times R$.

$\square$

Proposition 5.2.3. *Let $f : R \to R'$ be a ring homomorphism with I an ideal of R'. Suppose I and $f^{-1}(I)$ are regular ideals of R' and R, respectively. Then $R \bowtie^f I$ has Property A if and only if R and R' both have Property A.*

Proof. First we assume that both rings R and R' have Property A. Then by [DFF, Proposition 3.1] $tq(R \bowtie^f I) = tq(R) \times tq(R')$. By [H, Corollary 2.6], $tq(R)$ and $tq(R')$ then both have Property A and by Lemma 5.2.1, $tq(R \bowtie^f I) = tq(R) \times tq(R')$ has Property A. Now by [H, Corollary 2.6], $R \bowtie^f I$ has Property A.

We can essentially reverse this argument. Suppose that $R \bowtie^f I$ has Property A. Then by [H, Corollary 2.6] and [DFF, Proposition 3.1], $tq(R \bowtie^f I) = tq(R) \times tq(R')$ has Property A. By the lemma now $tq(R)$ and $tq(R')$ both have Property A, so finally by [H, Corollary 2.6], R and R' both have Property A.

$\square$

We note that this result may not hold when I and $f^{-1}(I)$ are not regular ideals. Consider the following counterexample.

Example 5.2.4. This example is adapted from [H, p. 174, Example 2]. Let K be an algebraically closed field and $D = K[X, Y]$. Let $\{P_\gamma\}$ be the set of all nonzero principal prime ideals of D, indexed by some set Γ and create the new index set $I := \Gamma \times \mathbb{N}$. For each index $i = (\gamma, n) \in I$, define $D_i = D/P_\gamma$ (so that for each γ we in fact take a countably infinite collection of copies of the same ring D/P_γ). We let $E = \prod_{i \in I} D_i$, and define $\phi : D \to E$ to be the canonical projection onto each coordinate (i.e., the "diagonal" mapping). Define the direct sum $J = \sum_{i \in I} D_i$ (note that J is an ideal of E that is not regular, and $\phi^{-1}(J) = 0$).

Now consider the ring $D \bowtie^{\phi} J$, which is canonically isomorphic to the $A+B$ construction (in the sense of [H, p. 169]), where $A = \phi(D)$, $B = \prod_{i \in I} D_i$. This ring does not have Property A (as described in the adapted example in [H]), even though the ring E does (this can be shown without much difficulty) and the domain D does trivially.

Lemma 5.2.5. *Let $R_1, ..., R_n$ be rings. Then $R_1 \times \cdots \times R_n$ has the (a.c.) if and only if each of the R_k has the (a.c.).*

Proof. As in Lemma 5.2.1 we only need to prove the lemma for two rings R, S. Suppose R, S have the (a.c.). Let $\overline{x}, \overline{y} \in R \times S$, say $\overline{x} = (x_1, x_2), \overline{y} = (y_1, y_2)$. Since R (resp. S) has the (a.c.) there exists $z_1 \in R$ (resp. $z_2 \in S$) with $Ann_R((x_1, y_1)) = Ann_R(z_1)$ (resp. $Ann_S((x_2, y_2)) = Ann_S(z_2)$). Let $\overline{z} = (z_1, z_2) \in R \times S$. Then straightforward calculations show that $Ann_{R \times S}(\overline{x}, \overline{y}) = Ann_{R \times S}(\overline{z})$, so $R \times S$ has the (a.c.).

Now suppose $R \times S$ has the (a.c.). We will show that R has the (a.c.); thus, let $x, y \in R$. We wish to find an element $z \in R$ such that $Ann_R((x, y)) = Ann(z)$. If $(x, y) = R$, then $Ann_R((x, y)) = 0 = Ann_R(1)$ and we are done. Otherwise $(x, y) \subset R$ (so $((x, 1), (y, 1)) \subset R \times S$) and by assumption, $Ann_{R \times S}(((x, 1), (y, 1))) = Ann_{R \times S}((a, b))$ for some $(a, b) \in R \times S$. Note that b must be regular: if there is some $z_2 \in S$ with $z_2 b = 0$, then $(0, z_2) \in Ann_{R \times S}((a, b)) = Ann_{R \times S}(((x, 1), (y, 1)))$ implies that $(0, z_2)(x, 1) = (0, z_2) = (0, 0)$, so $z_2 = 0$. We claim that $Ann_R((x, y)) = Ann_R(a)$.

Say $zx = zy = 0 \neq za$. Then $(z, 0)(x, 1) = (z, 0)(y, 1) = (0, 0) \neq (z, 0)(a, b) = (za, 0)$, contradicting our definition of $Ann((a, b))$. It follows that $Ann_R((x, y)) \subseteq Ann_R(a)$.

For the reverse inequality, say $za = 0$ for some $z \in R$. Let $\sum r_i x^{j_i} y^{k_i} \in (x, y)$. Note that $(z, 0) \in Ann_{R \times S}((a, b))$, and thus by assumption $(z, 0) \in Ann_{R \times S}((x, 1), (y, 1))$. In particular,

$$(z, 0) \sum (r_i, 1)(x, 1)^{j_i} (y, 1)^{k_i} = (0, 0),$$

so in the first coordinate here we have that $z \sum r_i x^{j_i} y^{k_i}$ is zero. Then $z \in Ann_R((x, y))$, so that $Ann_R(a) \subseteq Ann_R((x, y))$, giving equality. It follows that R has the (a.c.). By a symmetric argument we see that S has the (a.c.) as well.

□

Proposition 5.2.6. *Let $f : R \to R'$ be a homomorphism of rings and I an ideal of R' such that I, $f^{-1}(I)$ are regular ideals of R', R, respectively. Then $R \bowtie^f I$ has the (a.c.) if and only if R and R' both have the (a.c.).*

Proof. We note by [BDM, Corollary 2.5] that a ring R satisfies the (a.c.) if and only if $tq(R)$ does. With this and Lemma 5.2.5 the proof proceeds almost exactly as in Proposition 5.2.3, again noting that (under the given assumptions) $tq(R \bowtie^f I) = tq(R) \times tq(R')$. In fact each step is reversible: R, R' both have the (a.c.) $\Leftrightarrow$ $tq(R), tq(R')$ both have the (a.c.) $\Leftrightarrow$ $tq(R) \times tq(R') = tq(R \bowtie^f I)$ has the (a.c.) $\Leftrightarrow$ $R \bowtie^f I$ has the (a.c.).

□

Example 5.2.7. We will give an example to show why we assume I and $f^{-1}(I)$ to be regular for this proposition. For the appropriate counterexample, we create the same ring as in Example 5.2.4, except now we let $\{P_i\}$ be the set of all maximal ideals of D (this example is adapted from [H, p. 174, Example 1]).

As before, consider the ring $D \bowtie^\phi J$, which is canonically isomorphic to the $A + B$ construction, where $A = \phi(D)$, $B = \prod_{i \in I} D_i$. This ring does not have the (a.c.) (see the adapted example in [H]), even though it can be shown that the ring E does, and the domain D does trivially.

Lemma 5.2.8. *Let $R_1, ..., R_n$ be rings. Then $R_1 \times \cdots \times R_n$ is von Neumann regular if and only if each of the R_k is von Neumann regular.*

Corollary 5.2.9. *Let $R_1, ..., R_n$ be rings. Then $R_1 \times \cdots \times R_n$ is complemented if and only if each of the R_k is complemented.*

The proof of the lemma is trivial. The corollary follows directly from our definition of complemented rings and the fact that for any two rings R and S, $tq(R \times S) \cong tq(R) \times tq(S)$.

Proposition 5.2.10. *Let $f : R \to R'$ be a homomorphism of rings and I an ideal of R' such that I, $f^{-1}(I)$ are regular ideals of R', R, respectively. Then $R \bowtie^f I$ is complemented if and only if R and R' are both complemented.*

Proof. Suppose $R \bowtie^f I$ is complemented. Note by [DFF, Proposition 3.1] that $tq(R \bowtie^f I) = tq(R) \times tq(R') = tq(R \times R')$, so this ring is von Neumann regular by our assumption. But then $tq(R)$ and $tq(R')$ are von Neumann regular by Lemma 5.2.8; that is, R and R' are complemented.

Conversely, suppose R and R' are complemented. Then by Lemma 5.2.8, $tq(R) \times tq(R')$ is von Neumann regular, and thus $tq(R \bowtie^f I) = tq(R) \times tq(R')$ is von Neumann regular, so that $R \bowtie^f I$ is complemented.

□

Example 5.2.11. Complemented rings are always reduced, so we know by Theorem 5.1.1 that any complemented ring R must also satisfy the (a.c.). Note that $D \bowtie^f J$ of Example 5.2.7 is reduced but does not satisfy the (a.c.), so must not be complemented. Also, since domains are complemented, this gives an example of a ring $R \bowtie^f I$ that is not complemented even though R- being a domain- is complemented and R' is complemented (clearly, since R' is a product of fields).

Lemma 5.2.12. *Let R be a ring. Then R is complemented if and only if for every element $r \in R$ there is an element $s \in R$ satisfying that $rs = 0$ and $r + s$ is regular (in R).*

This lemma is well known, and a proof can be found in [DS3, Proposition 2.4]. The element s is called a *complement* for r (hence the terminology *complemented ring*), and is not necessarily unique.

Example 5.2.13. Now we give an example where R is complemented and neither of the ideals I or $f^{-1}(I)$ is regular, but $R \bowtie^f I$ is still complemented. Let E denote the direct sum of the fields $\mathbb{Z}/p\mathbb{Z}$ where the p run through all prime numbers in $\mathbb{Z}$, and let F denote the direct product of these fields. Define a homomorphism $\phi : \mathbb{Z} \to F$ as the diagonal mapping $\phi(n) = (\overline{n}, \overline{n}, \overline{n}, ...)$ (where $\overline{n}$ in each coordinate represents n reduced modulo p in the relevant field $\mathbb{Z}/p\mathbb{Z}$). We will look at the ring $\mathbb{Z} \bowtie^{\phi} E$. Note that $\mathbb{Z}$ is complemented (as is any domain), and that F is complemented being its own total quotient ring and von Neumann regular. It is clear that E is not regular in F. Also note that $f^{-1}(E) = 0$ is not regular in $\mathbb{Z}$. For a given element $\mathbf{e} \in E$ we will write e_p for its projection into (i.e., coordinate at) the summand $\mathbb{Z}/p\mathbb{Z}$.

Claim: An element $(r, \mathbf{r} + \mathbf{e})$ is a zero-divisor in $\mathbb{Z} \bowtie^{\phi} E$ if and only if $e_p \equiv -r(mod\ p)$ for some p.

If $r = 0$ then the statement is almost trivial. Since $\mathbf{e}$ is in the direct sum of fields, we can pick an index p where $e_p = 0$. Define the element $\mathbf{f}$ in this direct sum by $f_p = 1$, but $f_q = 0$ for $q \neq p$. Clearly then $(0, \mathbf{f})$ annihilates the element $(0, \mathbf{e})$. Now instead suppose $r \neq 0$. If at some index p, $e_p \equiv -r(mod\ p)$, then at this index $\mathbf{r} + \mathbf{e}$ is zero. Again take $\mathbf{f}$ as defined above, and $(0, \mathbf{f})$ annihilates the element $(r, \mathbf{r} + \mathbf{e})$.

Conversely, suppose that at no index p does $e_p \equiv -r(mod\ p)$. Then at no index is $\mathbf{r} + \mathbf{f}$ ever zero. By our defined (usual) multiplication on R, for an element $(s, \mathbf{s} + \mathbf{f})$ to annihilate $(r, \mathbf{r} + \mathbf{e})$, we would need $rs = 0$ in $\mathbb{Z}$. Note that r is nonzero by assumption, for if $r = 0$, then we would have $e_p \equiv -r(mod\ p)$ at some index, by definition of direct sum. Thus we must have that $s = 0$. Now for the element $(0, \mathbf{f})$ to annihilate $(r, \mathbf{r} + \mathbf{e})$ we need that at some index p, $f_p(r + e_p) \equiv 0(mod\ p)$. But $\mathbb{Z}/p\mathbb{Z}$ is a domain, and by assumption $(r + e_p)$ is nonzero here, so we must have $f_p = 0$. Since this is true for all p and since $s = 0$, then $(s, \mathbf{s} + \mathbf{f}) = (0, 0)$ and it follows that the annihilator of $(r, \mathbf{r} + \mathbf{e})$ is zero. This ends the proof of the claim.

Now we are ready to show that $\mathbb{Z} \bowtie^{\phi} E$ is complemented. Consider an element $(r, \mathbf{r} + \mathbf{e})$. Trivially, if this element is zero we take the identity as its complement and if it is regular we take zero as the complement. So we can assume this element is a nonzero zero-divisor. Thus by the claim, $e_p \equiv -r(mod\ p)$ for some p.

Let P be the set of prime indices p for which $e_p \equiv -r(mod\ p)$. Define $\mathbf{f}$ in the direct sum of fields $\mathbb{Z}/p\mathbb{Z}$ by $f_p = 1$ for $p \in P$ and $f_p = 0$ otherwise. We claim that $(0, \mathbf{f})$ is a complement for $(r, \mathbf{r} + \mathbf{e})$.

By construction $(0, f)\ (r, \mathbf{r} + \mathbf{e}) = (0, 0)$. Now we consider the sum, $(0, \mathbf{f}) + (r, \mathbf{r} + \mathbf{e}) = (r, \mathbf{r} + \mathbf{e} + \mathbf{f})$, which we wish to be regular. By the claim, it is sufficient to show that at no index p will we have $e_p + f_p \equiv -r (mod\ p)$. At any index $p \notin P$ we have that $f_p = 0$ and by definition of P, that $e_p \not\equiv -r (mod\ p)$, so $e_p + f_p \not\equiv -r (mod\ p)$. On the other hand, for $p \in P$ we have that $e_p \equiv -r (mod\ p)$, so that $e_p + f_p = e_p + 1 \not\equiv -r (mod\ p)$. Thus the sum satisfies the claim's condition to be regular, and so $(0, f)$ is a complement for $(r, \mathbf{r} + \mathbf{e})$. Then by Lemma 5.2.12, $\mathbb{Z} \bowtie^{\phi} E$ is complemented, so this gives our desired example.

5.3 tq(R) and Complemented Rings

If I is a regular ideal of a ring R, then $tq(R \bowtie I) \cong tq(R) \times tq(R)$. However, no general formula seems to be known for $tq(R \bowtie I)$ without the assumption that I be regular. It is not difficult to see that $tq(R^{\Delta}) \cong tq(R)^{\Delta}$ and that all regular elements of R^{Δ} are regular in the extension $R \bowtie I$. It follows that $tq(R^{\Delta}) \subseteq tq(R \bowtie I)$. However, there is no obvious guarantee that $tq(R \bowtie I) \subseteq tq(R \times R)$ in all cases. We will show that this in fact does hold, and use this knowledge to describe exactly when $R \bowtie I$ is complemented, and subsequently to describe the exact form of total quotient ring of $R \bowtie I$.

Lemma 5.3.1. *Let I be an ideal of a ring R. Then $Reg(R \bowtie I) \subseteq Reg(R \times R)$.*

Proof. We wish to show that every regular element of $R \bowtie I$ is a regular element of $R \times R$. We will show the contrapositive. That is, suppose $(r, r + i) \in R \bowtie I$ is a zero-divisor in $R \times R$; we will show that it is a zero-divisor in $R \bowtie I$. We can assume that $I \neq 0$ (the case $I = 0$ is trivial). If $r = 0$ (resp., $r + i = 0$) we can take the element $(j, 0)$ (resp., $(0, j)$) of $R \bowtie I$ to kill $(r, r + i)$ where j is any nonzero element of I. Thus we may assume that $r \neq 0$ and $r + i \neq 0$. Since $(r, r + i)$ is a zero-divisor in $R \times R$, there are elements $x, y \in R$ satisfying that $rx = 0$ and $(r + i)y = 0$. If $xy \neq 0$, then we can take the nonzero element

$(xy, xy) \in R \bowtie I$ to kill $(r, r+i)$. Assume instead that $xy = 0$. Note that it is possible that x or y is zero, but they cannot both be zero since (x, y) is nonzero by assumption. We now come to multiple cases, depending on whether x or y are elements of $Ann_R(I)$.

If $y \notin Ann_R(I)$, then we can pick any $j \in I$ such that $yj \neq 0$ and note that the nonzero element $(0, yj) \in R \bowtie I$ kills $(r, r+i)$. Similarly, if $x \notin Ann_R(I)$, we can find a $j \in I$ with $xj \neq 0$ and take the nonzero element $(xj, 0)$ to kill $(r, r+i)$.

Assume that $x \neq 0$. We can assume further that $x \in Ann_R(I)$ by the preceding paragraph. Then clearly the nonzero element $(x, x) \in R \bowtie I$ annihilates $(r, r+i)$. Finally, if $x = 0$ we must have $y \neq 0$. We can assume $y \in Ann_R(I)$, which gives $yi = 0$ so that $yr = y(r+i-i) = y(r+i) - yi = 0$, and we can take the nonzero element $(y, y) \in R \bowtie I$ to kill $(r, r+i)$. We have shown that $(r, r+i)$ is a zero-divisor in $R \bowtie I$; it follows that every regular element of $R \bowtie I$ is a regular element in $R \times R$.

□

The following proposition will in fact follow immediately from the upcoming Theorem 5.3.3. However we present it here as we have sufficient information to prove it without knowing the exact form of $tq(R \bowtie I)$. In this proposition we use the characterization that R is complemented if $tq(R)$ is von Neumann regular.

Proposition 5.3.2. *Let I be an ideal of a ring R. Then R is complemented if and only if $R \bowtie I$ is complemented.*

Proof. First note that $tq(R)^{\Delta} \cong tq(R^{\Delta})$ embeds into $tq(R \bowtie I)$, since the regular elements of $R (\cong R^{\Delta})$ embed into the regular elements of $R \bowtie I$. Note further that $tq(R \bowtie I)$ embeds into $tq(R \times R)$ via the canonical map $\frac{(r,r+i)}{(s,s+j)} \mapsto \frac{(r,r+i)}{(s,s+j)}$, since $Reg(R \bowtie I) \subseteq Reg(R \times R)$ by Lemma 5.3.1. Thus we have an inclusion of rings $tq(R)^{\Delta} \subseteq tq(R \bowtie I) \subseteq tq(R \times R) \cong tq(R) \times tq(R)$. Now by Lemma 3.2.1, we can view $tq(R \bowtie I)$ (up to isomorphism) as having the form $tq(R) \bowtie J$ for some ideal J of $tq(R)$. If we have proper containment $J \subset tq(R)$, then by [CM, Theorem 2.1], this ring $tq(R) \bowtie J$ is von Neumann regular if and only if $tq(R)$

is von Neumann regular; the case where $J = tq(R)$ has the same conclusion (Lemma 5.2.8), since in this case $tq(R) \bowtie J = tq(R) \bowtie tq(R) = tq(R) \times tq(R)$. Thus R is complemented if and only if $R \bowtie I$ is complemented.

□

Now we address the form of $tq(R \bowtie I)$ and show that it naturally occurs as a bowtie ring itself. In particular, we wish to show that $tq(R \bowtie I)$ will always be isomorphic to the ring $tq(R) \bowtie Itq(R)$. Since $Reg(R \bowtie I) \subseteq Reg(R \times R)$ we see by the proof of Lemma 5.3.1 that $tq(R \bowtie I) \cong tq(R) \bowtie J$ for some ideal J of $tq(R)$. But every localization of a ring is an epimorphism, so in particular the total quotient ring of a ring is an epimorphism. Thus, since $R \bowtie I \subseteq tq(R) \bowtie J$, we have by Theorem 4.2.12 that $J = Itq(R)$. We provide an alternate proof below (following some pieces of [D, Proposition 2.7(a)]), to allow the present section to be essentially self-contained.

Note that this generalizes the known property ([DF, Corollary 3.3(d)]) that $tq(R \bowtie I) \cong tq(R) \times tq(R)$ for any regular ideal I: If s is a regular element of I, then $s^{-1} \in tq(R)$ so that the ideal $Itq(R)$ of $tq(R)$ contains the element $ss^{-1} = 1$. Thus $Itq(R) = tq(R)$ and so $tq(R \bowtie I) \cong tq(R) \times tq(R) = tq(R) \bowtie tq(R) = tq(R) \bowtie Itq(R)$. It also generalizes the cases where I is zero or improper: If $I = 0$, then $R \bowtie 0 \cong R$ and $tq(R \bowtie 0) \cong tq(R) \cong tq(R) \bowtie 0$; if $I = R$, then $R \bowtie R = R \times R$ and $tq(R \bowtie R) = tq(R \times R) \cong tq(R) \times tq(R) = tq(R) \bowtie tq(R) = tq(R) \bowtie Rtq(R)$.

Theorem 5.3.3. *Let I be an ideal of a ring R. Then $tq(R \bowtie I) \cong tq(R) \bowtie Itq(R)$.*

Proof. Set $D := Reg(R \bowtie I)$. Then for each $(s, s+j) \in D$, $(s, s+j) \in Reg(R \times R)$ by Lemma 5.3.1, so that $s, s+j \in Reg(R)$. Thus s^{-1} and $(s+j)^{-1}$ both exist in $tq(R)$.

Now define the map $\phi : R \bowtie I \to tq(R) \bowtie Itq(R)$ by $\phi((s, s+j)) = (\frac{s}{1}, \frac{s}{1} + \frac{j}{1}) = (\frac{s}{1}, \frac{s+j}{1})$. Given $(s, s+j) \in D$ we wish now to show that $\phi((s, s+j))$ is a unit in $tq(R) \bowtie Itq(R)$. As we have seen, s^{-1} and $(s+j)^{-1}$ both exist in $tq(R)$, so that $(\frac{1}{s}, \frac{1}{s+j}) \in tq(R) \times tq(R)$. To see that it in fact lies in $tq(R) \bowtie Itq(R)$, note that $(\frac{1}{s}, \frac{1}{s+j}) = (\frac{1}{s}, \frac{1}{s} + \frac{1}{s+j} - \frac{1}{s}) =$

$(\frac{1}{s}, \frac{1}{s} - j(\frac{1}{s(s+j)})) \in tq(R) \bowtie Itq(R)$. It follows that every element of $\phi(D)$ is a unit in $tq(R) \bowtie Itq(R)$.

Now by the Universal Mapping Property of ring localizations, there exists a unique map $\Psi : tq(R \bowtie I) \to tq(R) \bowtie Itq(R)$ defined by $\Psi(\frac{(r,r+i)}{(s,s+j)}) = \frac{\phi((r,r+i))}{\phi((s,s+j))}$. We only need to show that this map is an isomorphism.

To show that it is onto, let $(\frac{r}{s}, \frac{r}{s} + \frac{i}{z}) \in tq(R) \bowtie Itq(R)$. Since s, z are both nonzero regular elements of R, we see that $sz \neq 0$ is also regular. Now we can take the element $\frac{(rz,rz+is)}{(sz,sz)}$ to map to $(\frac{r}{s}, \frac{r}{s} + \frac{i}{z})$ via Ψ. To show that the map is one-to-one, suppose that $\Psi(\frac{(r,r+i)}{(s,s+j)}) \equiv (0,0)$. Then it follows that $\frac{r}{s} \equiv 0$ and $\frac{r+i}{s+j} \equiv 0$ in $tq(R)$, and so we must have that $r = 0$, $r + i = 0$. Thus the kernel of Ψ is zero; that is, Ψ is one-to-one, and we conclude that $tq(R \bowtie I) \cong tq(R) \bowtie Itq(R)$.

□

Remark 5.3.4. The results in this section do not carry over easily to general bowtie rings. In fact, Lemma 5.3.1 often does not even hold. We will give two examples now. Let $f : R \to R'$ be a homomorphism of rings, and let I be an $f(R)-$subalgebra of R'. If I contains a regular element (of R') and $f^{-1}(I) = 0$, or if $I = 0$ and $f^{-1}(I)$ contains a regular element of R, then $Reg(R \bowtie^f I) \nsubseteq Reg(R \times R')$.

Proof. Suppose I contains the regular element i and that $f^{-1}(I) = 0$. Then $(0,i) \in R \bowtie^f I$. If $(0,i)(a,b) = (0,0)$ for some $(a,b) \in R \bowtie^f I$, then $b = 0$ (as i is regular), and so we must have that $a \in f^{-1}(I)$ (as $(a,0) \in R \bowtie^f I$); that is, $a = 0$. It follows that $(0,i)$ is regular in $R \bowtie^f I$. However, $(0,i)$ is a zero-divisor in $R \times R'$, annihilated by the element $(1,0)$.

If $I = 0$ but $f^{-1}(I)$ is regular, say with regular element j, then we employ a similar argument to above. We have that $(j,0) \in R \bowtie^f I$. If $(j,0)(a,b) = (0,0)$ then $a = 0$ from which it follows that $b \in I$; that is, $b = 0$, and so $(j,0)$ is regular in $R \bowtie^f I$. However $(j,0)$ is a zero-divisor in $R \times R'$, annihilated by the element $(0,1)$.

$\square$

Example 5.3.5. To show that each of the two situations described in the above remark is possible, we will provide explicit examples here. In the first case, let R be an integral domain, $R' := R[X]$, $I = XR[X]$, and $f : R \to R[X]$ the natural inclusion map. Clearly I contains a regular element (in fact, all of its nonzero elements). However $f^{-1}(I) = 0$ as no nonzero elements of R map to any polynomials with zero constant terms.

For the second case, let $R = \mathbb{Z}$, $R' = \mathbb{Z}/4\mathbb{Z}$, $I = 0$, and $f : R \to R'$ the canonical map. Note that $f^{-1}(I) = 4\mathbb{Z}$, which clearly contains regular elements of $\mathbb{Z}$. In each of these examples it follows by the above remark that $Reg(R \bowtie^f I) \nsubseteq Reg(R \times R')$.

Since regular elements may not embed into regular elements, we see now that without any extra assumptions on our bowtie ring $R \bowtie^f I$, there is no guarantee that $tq(R \bowtie^f I) \subseteq tq(R \times R')$, unlike in the simple bowtie ring case; in particular (defining $f_t : tq(R) \to tq(R')$ by $f_t(\frac{r}{s}) = \frac{f(r)}{f(s)}$) there is no guarantee that $tq(R \bowtie^f I)$ will have the form $tq(R) \bowtie^{f_t} Itq(R)$ as we might hope (or even the form $tq(R) \bowtie^{f_t} J$ for an arbitrary $f_t(tq(R))-$algebra J).

The explicit form of the total quotient ring found for simple bowtie rings will be vital in the next chapter, where we will use it to find the integral closure of a simple bowtie ring (Corollary 6.1.9).

Chapter 6: Integrality

6.1 Integrality and Integral Closure

In this chapter we will study integrality and related properties, such as lying-over, going-up, going-down, and normal pairs as they apply to extensions of bowtie rings. For a given bowtie ring extension $R \bowtie I \subset R \bowtie J$ (with $I \subset J$ ideals of a ring R) it follows from [D, Remark 1] that $R \subset R \bowtie J$ is an integral extension and thus so is $R \bowtie I \subset R \bowtie J$ since $R \bowtie I$ is an intermediate ring (for reference we will also offer this as a corollary to the first theorem of this chapter).

We construct the following theorem on integrality with our most general form of an extension of bowtie rings. The details relevant to us will be presented in the corollaries that follow. The proof of this theorem is essentially a generalization of [DFF, Lemma 3.6].

Theorem 6.1.1. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, I an $f(R)-$subalgebra of T', and J an $f(T)-$subalgebra of T' with $I \subseteq J$. Then $T \bowtie^f J$ is integral over $R \bowtie^f I$ if and only if the ring extensions $R \subseteq T$ and $f(R) + I \subseteq f(T) + J$ are both integral.*

Proof. Suppose $T \bowtie^f J$ is integral over $R \bowtie^f I$. For any element $t \in T$, we can find a monic polynomial $p(X)$ over $R \bowtie^f I$ satisfied by $(t, f(t))$ and clearly t satisfies this same polynomial with coefficients projected to the first coordinate. Similarly, for any $f(t) + j \in f(t) + J$ we know that $(t, f(t) + j)$ satisfies a monic $q(X)$ over $R \bowtie^f I$, so $f(t) + j$ satisfies this same monic with coefficients projected to the second coordinate.

Conversely, suppose that both of the ring extensions $R \subseteq T$ and $f(R)+I \subseteq f(T)+J$ are integral. Let $(t, f(t) + j) \in T \bowtie^f J$. By assumption, there are coefficients a_k and $f(b_h) + i_h$

in R and $f(R)+I$, respectively, such that

$$(t)^n + a_{n-1}(t)^{n-1} + \cdots + a_0 = 0$$

and

$$(f(t)+j)^m + (f(b_{m-1}) + i_{m-1})(f(t)+j)^{m-1} + \cdots + f(b_0) + i_0 = 0.$$

Now, if $(t, f(t)+j)^{n+m} = (0,0)$ then the element $(t, f(t)+j)$ is nilpotent, and integrality follows trivially. Otherwise, it will satisfy the monic polynomial $p(X)q(X)$ over $T \bowtie^f J$, where

$$p(X) = (X^n + (a_{n-1}, f(a_{n-1}))X^{n-1} + \cdots + (a_0, f(a_0)))$$

and

$$q(X) = (X^m + (b_{m-1}, f(b_{m-1}) + i_{m-1})X^{m-1} + \cdots + (b_0, f(b_0) + i_0).$$

In either case it follows that the extension $R \bowtie^f I \subseteq T \bowtie^f J$ is integral.

□

Corollary 6.1.2. *Let $R \subset T$ be rings with ideals I, J, respectively such that $I \subseteq J$. Then $R \bowtie I \subset T \bowtie J$ is integral if and only if $R \subset T$ is integral.*

Corollary 6.1.3. *If $I \subset J$ are ideals of a ring R, then the extension $R \bowtie I \subset R \bowtie J$ is integral. Thus (by [K, Theorem 43]) it satisfies lying-over, going-up, and incomparability (as defined in the next section).*

Corollary 6.1.4. *Let $f : R \to R'$ be a ring homomorphism, and let I, J be $f(R)-$subalgebras of R' with $I \subset J$. If $J \subseteq f(R)$ or if $J \subseteq Nil(R')$ then the extension $R \bowtie^f I \subset R \bowtie^f J$ is integral.*

Proof. Suppose that $J \subseteq f(R)$. Let $f(a) + j \in f(R) + J$ and note that $f(a) \in f(R) + I$ trivially, and that $j \in f(R) + I$ by assumption. Thus $f(R) + J$ is integral over $f(R) + I$ and so by the Theorem 6.1.1, $R \bowtie^f I \subset R \bowtie^f J$ is integral.

The case where $J \subseteq Nil(R')$ is similar except that we now note that $f(R)+J$ is integral over $f(R)+I$ since every element of $f(R)$ (being in $f(R)+I$) is integral over it as is every element in J (trivially, as each element is nilpotent). Thus every element of $f(R)+J$ is the sum of two integral elements over $f(R)+I$, so is integral over $f(R)+I$, and again by the Theorem 6.1.1, $R \bowtie^f I \subset R \bowtie^f J$ is integral.

□

Recall that given a ring extension $R \subseteq T$ we use $\overline{R}^T$, to denote the integral closure of R in T, and simply $\overline{R}$ to denote the integral closure of R (that is, the integral closure of R in its total quotient ring). Following [HS], we say that an element $r \in R$ is *integral* over an ideal I of R, if there exists an integer n and coefficients $i_k \in I^k$ with $k = 1, ..., n$ such that $r^n + i_1 r^{n-1} + \cdots + i_n = 0$. The set of all such elements of R is called the *integral closure of I in R* and denoted by $\overline{I}$. This set in fact forms an ideal of R [HS, Corollary 1.3.1]. We say that an ideal J of R is *integral* over I if $J \subseteq \overline{I}$.

Proposition 6.1.5. *Let $f : R \to R'$ be a ring homomorphism. Let $I \subset J$ be ideals of R' with J integral over I, i.e., $J \subseteq \overline{I}$. Then the extension $R \bowtie^f I \subset R \bowtie^f J$ is integral. In particular, the extension $R \bowtie^f I \subset R \bowtie^f \overline{I}$ is integral.*

Proof. Suppose that J is integral over I. Let $(a, f(a)+j) \in R \bowtie^f J$. By assumption there exist coefficients i_k in $I^k \subseteq I$ with $j^n + i_1 j^{n-1} + \cdots + i_n = 0$. But then $(0, j^n) + (0, i_1)(0, j^{n-1}) + \cdots + (0, i_n) = (0, 0)$, showing that $(0, j)$ is integral over $R \bowtie^f I$. Also, the element $(a, f(a))$ of $R \bowtie^f I$ is trivially integral over it. Then $(a, f(a)+j) = (a, f(a)) + (0, j)$ is integral over $R \bowtie^f I$, being the sum of two integral elements. □

We note that the preceding proposition could also be stated as a corollary, with a slightly modified proof. Since each $f(a) \in f(R)$ lies in $f(R)+I$, $f(a)$ is trivially integral over this ring. By assumption each $j \in J$ is integral over I and so it is easy to see that j is integral over the ring $f(R)+I$. It follows that the ring $f(R)+J$ is integral over $f(R)+I$ and so by Theorem 6.1.1 above, the extension $R \bowtie^f I \subset R \bowtie^f J$ is integral.

Theorem 6.1.6. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, I an $f(R)-$subalgebra of T', and J an $f(T)-$subalgebra of T' with $I \subseteq J$. Then the integral closure of $R \bowtie^f I$ in $T \bowtie^f J$ is $\overline{R}^T \bowtie^f (J \cap \overline{f(R) + I}^{T'})$.*

Proof. Let $(t, f(t) + j) \in \overline{R}^T \bowtie^f (J \cap \overline{f(R) + I}^{T'})$. Then t is integral over R and it follows easily that $(t, f(t))$ is integral over $R \bowtie^f I$: if $t^n + r_{n-1}t^{n-1} + \cdots + r_0 = 0$ ($r_k \in R$), then $(t, f(t))^n + (r_{n-1}, f(r_{n-1})(t, f(t))^{n-1} + \cdots + (r_0, f(r_0)) = (0, 0)$. Also, we are assuming that j is integral over $f(R) + I$ by some monic, say

$$j^n + (f(s_{n-1}) + i_{n-1})j^{n-1} + \cdots + (f(s_0) + i_0) = 0.$$

with each $s_k \in R, i_k \in I$. But then $(0, j)$ is integral over $R \bowtie^f I$, satisfying the monic

$$X[X^n + (s_{n-1}, f(s_{n-1}) + i_{n-1})X^{n-1} + \cdots + (s_0, f(s_0) + i_0)].$$

Then $(t, f(t))$ and $(0, j)$ are both in the integral closure of $R \bowtie^f I$ (in $T \bowtie^f J$), and thus so is their sum, $(t, f(t) + j)$.

Now we turn to the reverse inclusion. Let $(t, f(t) + j)$ belong to the integral closure of $R \bowtie^f I$ in $T \bowtie^f J$. We first claim that t is integral over R. Suppose

$$(t, f(t) + j)^n + (r_{n-1}, f(r_{n-1}) + i_{n-1})(t, f(t) + j)^{n-1} + \cdots + (r_0, f(r_0) + i_0) = (0, 0)$$

Then $t^n + r_{n-1}t^{n-1} + \cdots + r_0 = 0$ so we see that $t \in \overline{R}^T$. Since f is a homomorphism it follows further that $f(t)$ is integral over $f(R)$, and thus over the ring $f(R) + I$. Now from the above equation we also have that

$$(f(t) + j)^n + (f(r_{n-1}) + i_{n-1})(f(t) + j)^{n-1} + \cdots + (f(r_0) + i_0) = 0,$$

so that $f(t) + j$ is integral over $f(R) + I$, and since $f(t)$ is integral over this ring so is

$j = (f(t)+j) - f(t)$. That is, $j \in \overline{f(R)+I}^{T'}$, and since $j \in J$ by assumption, we have that $(t, f(t)+j) \in \overline{R}^T \bowtie^f (J \cap \overline{f(R)+I}^{T'})$.

$\square$

We note that there are multiple ways to write the integral closure in Theorem 6.1.6. Since the integral closure of $f(R)+I$ in T' is intersected with J, the integral closure $\overline{f(R)+I}$ could be taken in any subring of T' containing J. In particular, we could consider the smallest intermediate ring between $f(R)+I$ and T' containing J (namely the ring $f(R)+J$), and write the integral closure of $R \bowtie^f I$ in $T \bowtie^f J$ as $\overline{R}^T \bowtie^f (J \cap \overline{f(R)+I}^{f(R)+J})$. The way that we recorded the integral closure in Theorem 6.1.6 was merely for brevity and simplicity.

Corollary 6.1.7. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, and J an $f(T)-$subalgebra of T'. Then the integral closure of $\Gamma(f_R) := R \bowtie^f 0$ in $T \bowtie^f J$ is $\overline{R}^T \bowtie^f (J \cap \overline{f(R)}^{T'})$.*

Corollary 6.1.8. *Let $R \subseteq T$ be an extension of rings with ideals $I \subseteq J$, respectively. Then the integral closure of $R \bowtie I$ in $T \bowtie J$ is $\overline{R}^T \bowtie (J \cap \overline{R}^T)$. In particular, identifying R with its image along the diagonal in $T \bowtie J$ (that is, $R \cong R^\Delta = R \bowtie 0$), then the integral closure of R in $T \bowtie J$ is $\overline{R}^T \bowtie (J \cap \overline{R}^T)$.*

With Theorem 6.1.6 and the subsequent Corollary 6.1.8, we are now well-equipped to determine the integral closure of any simple bowtie ring $R \bowtie I$ (that is, the integral closure of $R \bowtie I$ in its total quotient ring). It is known that if I is a regular ideal, then the integral closure of $R \bowtie I$ is $\overline{R} \bowtie \overline{R} = \overline{R} \times \overline{R}$ ([DF, Corollary 3.3 (c) and (d)]). Since $Itq(R) = tq(R)$ for any regular ideal I of R, this result is generalized by the following corollary.

Corollary 6.1.9. *Let I be an ideal of a ring R. Then the integral closure of $R \bowtie I$ is $\overline{R} \bowtie (Itq(R) \cap \overline{R})$.*

Proof. By Theorem 5.3.3, the total quotient ring of $R \bowtie I$ is $tq(R) \bowtie Itq(R)$. Since this is a bowtie ring containing $R \bowtie I$ we can now apply Corollary 6.1.8 (setting $T = tq(R)$, $J =$

$Itq(R))$, and find that $\overline{R \bowtie I} = \overline{R \bowtie I}^{tq(R) \bowtie Itq(R)} = \overline{R} \bowtie (Itq(R) \cap \overline{R})$.

□

We now apply the above results to determine when a bowtie ring is integrally closed inside a (bowtie ring) extension ring.

Theorem 6.1.10. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, I an $f(R)-$subalgebra of T', and J an $f(T)-$subalgebra of T' with $I \subseteq J$. Then $R \bowtie^f I$ is integrally closed in $T \bowtie^f J$ if and only if the following conditions hold:*

1. *R is integrally closed in T,*
2. *$f(R) + I$ is integrally closed in $f(R) + J$, and*
3. *$J \cap (f(R) + I) = I$.*

Proof. If the three conditions are satisfied then clearly $R \bowtie^f I$ is integrally closed in $T \bowtie^f J$ by Theorem 6.1.6 (and the remark following it).

Conversely, suppose that $R \bowtie^f I$ is integrally closed in $T \bowtie^f J$. Clearly R must be integrally closed in T, by Theorem 6.1.6. Now suppose that some $f(r) + j \in f(R) + J$ satisfies a monic over $f(R) + I$, say $(f(r) + j)^n + (f(r_{n-1}) + i_{n-1}) + \cdots + (f(r_0) + i_0) = 0$. Then $(r, f(r) + j)^n + (r_{n-1}, f(r_{n-1}) + i_{n-1})(r, f(r) + j)^{n-1} + \cdots + (r_0, f(r_0) + i_0) = (0, 0)$, and since $R \bowtie^f I$ is integrally closed in $T \bowtie^f J$ we must have that $(r, f(r) + j) \in R \bowtie^f I$, so $f(r) + j \in f(R) + I$. Finally, the requirement that $J \cap (f(R) + I) = I$ follows from the previous two conditions with Theorem 6.1.6 and Lemma 2.1.3.

□

Corollary 6.1.11. *Let $R \subseteq T$ be a ring extension with ideals $I \subseteq J$, respectively. Then $R \bowtie I$ is integrally closed in $T \bowtie J$ if and only if R is integrally closed in T and $J \cap R = I$.*

Note here the significant difference between bowtie extensions where we leave the ring fixed and those where we leave the ideal fixed. In the case above, where the ideal is essentially kept fixed, we can construct an extension that is not integral. In fact, we will see in the next section that if the ideal J is common to R and T we can use them to create a ring extension $R \bowtie J \subset T \bowtie J$ in which every intermediate ring is integrally closed in $T \bowtie J$, in particular a situation where $R \bowtie J$ is integrally closed in $T \bowtie J$. This greatly contrasts keeping the ring R fixed, as $R \bowtie I \subset R \bowtie J$ is always necessarily integral (Corollary 6.1.3).

Recall that by saying a ring R is *integrally closed* we mean that R is integrally closed in its total quotient ring. Now we have an immediate application of Corollary 6.1.11, keeping in mind Theorem 5.3.3.

Corollary 6.1.12. *Let I be an ideal of a ring R. Then $R \bowtie I$ is integrally closed if and only if R is integrally closed and $Itq(R) \cap R = I$.*

Corollary 6.1.13. *If R is a total quotient ring (i.e., $R = tq(R)$) and I is an ideal of R, then $R \bowtie I$ is integrally closed if and only if R is integrally closed.*

If I is a proper regular ideal, then $Itq(R) = tq(R) \neq I$. Thus we see that $R \bowtie I$ is never integrally closed for any proper regular ideal I of R. This information also follows from [DF, Corollary 3.3(c), (d)]. Thus the assumption that $I \subseteq Z(R)$ in the following corollary is justified.

Corollary 6.1.14. *If $I \subseteq Z(R)$ and I is prime, then $R \bowtie I$ is integrally closed if and only if R is integrally closed.*

Proof. Assume that $I \subseteq Z(R)$. If $R \bowtie I$ is integrally closed (by which we mean integrally closed in $tq(R) \bowtie Itq(R)$ by Theorem 5.3.3), then R is integrally closed, by Corollary 6.1.11. For the converse, assume that R is integrally closed and note that $Itq(R) \cap R = \{r \in R | dr \in I \text{ for some } d \in Reg(R)\}$, which contains I. Let $r \notin I$ be an element of $Itq(R) \cap R$. Then there is some $d \in Reg(R)$ satisfying that $dr \in I$. Since I is prime and $r \notin I$ we must have that I contains the regular element r, which is absurd.

□

Some texts use the term *minimal prime* of a domain to describe a prime that is minimal over zero. We use it in the natural sense that it contains no other primes properly. Thus in a domain the only minimal prime is the zero ideal. By [K, Theorem 84] the set of zero-divisors of a ring R contains the union of all the minimal primes of R.

Corollary 6.1.15. *If P is a minimal prime ideal of a ring R, then $R \bowtie P$ is integrally closed if and only if R is integrally closed.*

6.2 Normal Pairs

Given a ring extension $R \subset T$, we often refer to the ordered pair (R, T) of rings simply as a *pair*. Additionally, we say that (R, T) is a *normal pair* if S is integrally closed in T for each intermediate ring $R \subseteq S \subseteq T$. Recall that a ring R is called *complemented* if $tq(R)$ is von Neumann regular. The more difficult direction of the upcoming theorem was proved by Dobbs and Shapiro in [DS3, Proposition 4.5]. For convenience, we first we present a lemma describing the easier direction. We will prove this lemma for general bowtie rings, though will return to simple bowtie rings for the proposition.

Lemma 6.2.1. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, I an $f(R)-$subalgebra of T', and J an $f(T)-$subalgebra of T' with $I \subseteq J$. If $(R \bowtie^f I, T \bowtie^f J)$ is a normal pair, then (R, T) is a normal pair.*

Proof. Suppose $(R \bowtie^f I, T \bowtie^f J)$ is a normal pair. If (R, T) is not a normal pair, then there is some intermediate ring S (possibly R itself) and some element $t \in T \setminus S$ which is integral over S. Say

$$t^n + s_{n-1}t^{n-1} + \cdots + s_1 t_1 + s_0 = 0.$$

Note that $S \bowtie^f J := \{(s, f(s) + j) | s \in S, j \in J\}$ is a ring lying between $R \bowtie^f I$ and $T \bowtie^f J$. Further, the element $(t, f(t)) \in T \bowtie^f J \setminus S \bowtie^f J$ satisfies the monic

$$X^n + (s_{n-1}, f(s_{n-1})) + \cdots + (s_1, f(s_1))X + (s_0, f(s_0)),$$

contradicting that $S \bowtie^f J$ is (by assumption) integrally closed in $T \bowtie^f J$.

□

The converse of this lemma is not true in general, even for an extension of simple bowtie rings. For instance, consider the rings $R = \mathbb{Z}_{2\mathbb{Z}}$ and $T = \mathbb{Q}$. Take the unique maximal ideal $I = 2R$ of R and the improper ideal $J = \mathbb{Q}$ of T, so that $T \bowtie J = \mathbb{Q} \bowtie \mathbb{Q} = \mathbb{Q} \times \mathbb{Q}$. We see easily that (R, T) is a normal pair, since it is an integrally closed minimal extension. However, $(R \bowtie I, T \bowtie J)$ is not a normal pair. Indeed, $R \bowtie I$ is not even integrally closed in $T \bowtie J$; by Corollary 6.1.8, the integral closure of $R \bowtie I$ (in $T \bowtie J$) is $\mathbb{Z}_{2\mathbb{Z}} \bowtie \mathbb{Z}_{2\mathbb{Z}} = \mathbb{Z}_{2\mathbb{Z}} \times \mathbb{Z}_{2\mathbb{Z}}$.

Proposition 6.2.2. *Let $R \subset T$ be a ring extension with ideals I, J respectively, such that $J \cap R = I$, and R is complemented. Then $(R \bowtie I, T \bowtie J)$ is a normal pair if and only if (R, T) is a normal pair.*

Proof. With the given conditions, (R, T) being a normal pair implies that $(R \bowtie I, T \bowtie J)$ is a normal pair by [DS3, Proposition 4.5]. The converse follows from Lemma 6.2.1.

□

We record an unexpected corollary before moving on. We show that given the conditions of the above proposition, every ideal J of T contains exactly one ideal over $J \cap R$ of each intermediate ring $R \subseteq S \subseteq T$.

Corollary 6.2.3. *Let (R, T) be a normal pair such that R is complemented, and let J be an ideal of T. Set $I := J \cap R$. Then for every intermediate ring $R \subseteq S \subseteq T$, there is exactly one ideal K of S satisfying the set inclusions $I \subseteq K \subseteq J$, namely $K = J \cap S$ (and necessarily $K = IS$). Consequently, no nonzero ideal J of T intersects R trivially.*

Proof. Let $R \subseteq S \subseteq T$ as unital subrings. Let K be an ideal of S satisfying $I \subseteq K \subseteq J$, and note that $S \bowtie K$ is a ring lying between $R \bowtie I$ and $T \bowtie J$. Since R is complemented it follows by Proposition 6.2.2 that $(R \bowtie I, T \bowtie J)$ is a normal pair; in particular, $S \bowtie K$

is integrally closed in $T \bowtie J$. Thus by Corollary 6.1.11, $K = J \cap S$. Since IS is an ideal of S satisfying $I \subseteq IS \subseteq J$ the same reasoning shows that $IS = J \cap S = K$.

Finally, if $J \cap R = 0$, then we have for every intermediate ring $R \subseteq S \subseteq T$ that there is exactly one ideal $K := J \cap S$ of S satisfying $0 \subseteq K \subseteq J$. Since the ideals 0 and J of the ring $S = T$ clearly both satisfy this assumption we must have that $J = 0$.

□

Another way to see the last statement of this corollary, is to note that the extension $R \bowtie 0 \subset T \bowtie J$ will contain the intermediate ring $T \bowtie 0$, and $T \bowtie J$ is integral over this ring (Corollary 6.1.3), so that (R, T) cannot be a normal pair, by Proposition 6.2.2.

For a trivial example of how to apply this corollary, note that the ring $R = \mathbb{Z}$ is integrally closed in the extension ring $T = \mathbb{Z}[X]$. There are many simple ways to see that (R, T) is not a normal pair, but we could also cite the above corollary, taking for instance the ideal $J = (X)$ in T. Then $J \cap R = 0$, but J intersects many intermediate rings (in fact, every intermediate ring) nontrivially, so by the corollary, (R, T) is not a normal pair.

Let us make one quick note regarding the converse of this corollary. That is, one may wonder whether the condition on the ideals of T is equivalent to (R, T) being a normal pair. We answer this quickly in the negative. Simply take any finite extension of fields, say $R = \mathbb{Q} \subset T = \mathbb{Q}(\sqrt{2})$. Then there is only one ideal of T to check, namely $J = 0$. Clearly J intersects every intermediate ring trivially, so that the ideals of T satisfy the conditions of the corollary. Further, R is complemented, obviously, since it is a field. However (R, T) is not a normal pair. On the contrary, T is an algebraic field extension of R, so in particular, R is not algebraically closed (thus, since R is a field, not integrally closed) in T. For an example where the rings are not fields, or even domains, simply replace this R and T with R^n and T^n, respectively, for any natural number $n \geq 2$.

We now return to Proposition 6.2.2. If we remove the assumption that $J \cap R = I$ we run into problems by Corollary 6.1.8, since if $J \cap R \neq I$ then $R \bowtie I$ itself is not even integrally closed in $T \bowtie J$. If it were, we would need R to be integrally closed in T, and so by Corollary 6.1.8 the integral closure of $R \bowtie I$ (in $T \bowtie J$) would be $\overline{R}^T \bowtie (J \cap \overline{R}^T) = R \bowtie (J \cap R) \supset R \bowtie I$ (cf. the example following Lemma 6.2.1).

Thus the assumption that $J \cap R = I$ is indispensable in constructing a normal pair $(R \bowtie I, T \bowtie J)$. On the other hand, if we refine this assumption by requiring that $J = I$, we are able to strengthen the result by removing the restriction that R be complemented. In fact we can give this result in the context of general bowtie rings.

Theorem 6.2.4. *Let $R \subseteq T$ be a ring extension, $f : T \to T'$ a ring homomorphism, and J an $f(T)-$subalgebra of T'. Then $(R \bowtie^f J, T \bowtie^f J)$ is a normal pair if and only if (R, T) is a normal pair.*

Proof. By Lemma 3.2.13, every ring between $R \bowtie^f J$ and $T \bowtie^f J$ is of the form $S \bowtie^f J$ for some ring $R \subseteq S \subseteq T$. Assume that (R, T) is a normal pair, and thus for each intermediate ring S, S is integrally closed in T. Then by Theorem 6.1.6 and the comments following it, we can see that the integral closure of $S \bowtie^f J$ in $T \bowtie^f J$ is $\overline{S}^T \bowtie^f (J \cap \overline{f(R) + J}^{f(R)+J}) = S \bowtie^f (J \cap (f(R) + J)) = S \bowtie^f J$, so that $S \bowtie^f J$ is integrally closed in $T \bowtie^f J$. Thus $(R \bowtie^f J, T \bowtie^f J)$ is a normal pair.

The converse follows immediately from Lemma 6.2.1.

□

6.3 LO, INC, GU, and GD

In this section we study some properties or ring extension that are closely related to integrality. We will describe these properties briefly before proceeding. Recall that for a ring extension $R \subset T$, we say that a prime $Q \in Spec(T)$ *lies over* $P \in Spec(R)$ (or *contracts to P in R*) if $Q \cap R = P$.

Definition 6.3.1. Let $R \subset T$ be a ring extension. We say that the extension satisfies *lying-over (LO)* if for every $P \in Spec(R)$, there exists a $Q \in Spec(T)$ lying over P. The extension satisfies *incomparability (INC)* if every pair of primes $Q_0, Q_1 \in Spec(T)$ lying over the same prime in R are incomparable; alternatively, whenever $Q_0 \subseteq Q_1 \in Spec(T)$ lie over $P \in Spec(R)$ we have $Q_0 = Q_1$. The extension satisfies *going-up (GU)* if whenever $P_0 \subseteq P_1 \in Spec(R)$ and $Q_0 \in Spec(T)$ lies over P_0, there exists a prime Q_1 of T lying over P_1 with $Q_0 \subseteq Q_1$. The extension satisfies *going-down (GD)* if whenever $P_0 \subseteq P_1 \in Spec(R)$ and $Q_1 \in Spec(T)$ lies over P_1, there exists a $Q_0 \in Spec(T)$ lying over P_0 with $Q_0 \subseteq Q_1$.

We have seen in Corollary 6.1.3 that any extension of the form $R \bowtie I \subset R \bowtie J$ will satisfy lying-over, going-up, and incomparability. In this section we will show that any extension of the form $R \bowtie I \subset T \bowtie J$ satisfies lying-over (resp., incomparability, going-up) exactly when $R \subset T$ satisfies lying-over (resp., incomparability, going-up). As we will see however, the conditions for a ring extension of the form $R \bowtie I \subset T \bowtie J$ (or even $R \bowtie I \subset R \bowtie J$) to satisfy going-down become much more complicated. The results of this section do not generalize well to the context of general bowtie rings; to avoid requiring myriad conditions and much longer proofs in all of these results, we will adhere to the case of simple bowtie rings for the remainder of the chapter.

Proposition 6.3.2. *Let $R \subset T$ be rings with ideals I, J, respectively, such that $I \subseteq J$. Then the extension $R \bowtie I \subset T \bowtie J$ satisfies lying-over if and only if the extension $R \subset T$ satisfies lying-over.*

Proof. First suppose that $R \subset T$ satisfies lying-over. Let $P_0 \in Spec(R \bowtie I)$. Without loss of generality we can assume that P_0 has the form $P_0 = \{(p, p+i) | p \in P, i \in I\}$ for some $P \in Spec(R)$ (by Lemma 2.1.1 this is one of only two possible forms for P_0; the case for the second form will follow by a virtually identical argument). By assumption we can find a prime $Q_0 \in Spec(T)$ contracting to P in R. Then by Lemmas 1.5.8 and 2.1.4, the set $Q_0 := \{(q, q+j) | q \in Q, j \in J\}$ is a prime ideal of $T \bowtie J$ and contracts to P_0.

Conversely, suppose that $R \bowtie I \subset T \bowtie J$ satisfies lying-over, and let $P \in Spec(R)$. Consider the prime $P_0 := \{(p, p+i) | p \in P, i \in I\}$ of $R \bowtie I$. By assumption some $Q_0 \in Spec(T \bowtie J)$ lies over it. If Q_0 is of the first form of 2.1.1, say $Q = \{(q, q+j) | q \in Q, j \in J\}$ for some $Q \in Spec(T)$, then by Lemma 2.1.4 it is easy to see that $Q \cap R = P$. Thus

$Q \in Spec(T)$ lies over P.

Finally, if Q_0 is of the second form of Lemma 2.1.1, say $Q_0 = \{(q, q+j)|q \in Q, j \in J\}$ for some $Q \in Spec(T)$, then by Lemma 2.1.4 its contraction to $R \bowtie I$ must look like $\{(q'+i, q')|q' \in Q \cap R, i \in I\}$. Then by assumption we have

$$\{(p, p+i)|p \in P, i \in I\} = \left\{(q'+i, q')|q' \in Q \cap R, i \in I\right\}.$$

By comparing the second coordinates we see that every $p = p + 0 \in P$ must also be in $Q \cap R$. By comparing the first coordinates we see that every $q' = q' + 0$ in $Q \cap R$ must be in P. Thus $Q \cap R = P$, so Q lies over P.

□

Lemma 6.3.3. *[Krull] For a ring R, $Nil(R) = \bigcap P$ where the intersection runs over all prime ideals P of R.*

As in [DS5] we will say that a ring R satisfies *ULO* if every ring extension $R \subset T$ satisfies lying-over. If R is a zero-dimensional ring then R satisfies ULO (cf. [K, Section 1.6, Exercise 2]), and further $R \bowtie I$ is zero-dimensional ([D, Remark 2.1]) so also satisfies ULO.

Proposition 6.3.4. *Let I be an ideal of a ring R. If $R \bowtie I$ satisfies ULO then R satsifies ULO. Conversely, if R satisfies ULO and $I \subseteq Nil(R)$ then $R \bowtie I$ satisfies ULO.*

Proof. First suppose that $R \bowtie I$ satisfies ULO. Let $R \subset S$ be a ring extension and $P \in Spec(R)$. Note that $S \bowtie IS$ is a ring extension of $R \bowtie I$, and $P' = \{(p, p+i)|p \in P, i \in I\}$ is a prime of $R \bowtie I$, so by assumption some $Q' \in Spec(S \bowtie IS)$ lies over P'. Now $Q' \cap (R \bowtie I)$ is a prime of $R \bowtie I$, and has the form $\{(q, q+i)|q \in Q \cap R, i \in I\}$ or $\{(q+i, q)|q \in Q \cap R, i \in I\}$ for some $Q \in Spec(S)$. In the first case we conclude that $P = Q \cap R$ immediately. In the second case, we have that $\{(p, p+i)|p \in P, i \in I\} = \{(q+i, q)|q \in Q \cap R, i \in I\}$; setting all the i to zero in the second coordinate gives that

$P \subseteq Q \cap R$ and doing so in the first coordinate gives that $Q \cap R \subseteq P$. It follows that $Q \in Spec(S)$ lies over P.

For the converse, under the assumption $I \subseteq Nil(R)$, we suppose that R satisfies ULO. Let $R \bowtie I \subset S$ be a ring extension. Identifying R with R^{Δ} in $R \times R$ (as usual) we see that $R^{\Delta} \subset S$ is a ring extension, so satisfies lying-over by assumption. Now let $P_0 \in Spec(R \bowtie I)$. Since $I \subset Nil(R)$, it follows by Lemmas 1.5.8 and 6.3.3 that we can write $P_0 = \{(p, p+i) | p \in P, i \in I\}$ for some prime P of R, and further, P_0 is the unique prime of $R \bowtie I$ lying over P. By assumption we can find a $Q \in Spec(S)$ lying over P, and it follows that $Q \cap (R \bowtie I) = P_0$, so that $R \bowtie I \subset S$ satisfies lying-over.

□

Proposition 6.3.5. *Let $R \subseteq T$ be rings with ideals I, J, respectively, such that $I \subseteq J$. Then the extension $R \bowtie I \subseteq T \bowtie J$ satisfies INC if and only if the extension $R \subseteq T$ satisfies INC.*

Proof. Suppose $R \bowtie I \subset T \bowtie J$ satisfies INC. Let $Q \subseteq Q'$ be primes of T contracting to the same prime P in R. Define the sets $Q_0 =$
$\{(q, q+j) | q \in Q, j \in J\}$, $Q_1 = \{(q', q'+j\}$, and $P_0 = \{(p, p+i) | p \in P, i \in I\}$. Then by Lemma 1.5.8, $Q_0, Q_1 \in Spec(T \bowtie J)$ and $P_0 \in Spec(R \bowtie I)$, and by Lemma 2.1.4 both Q_0 and Q_1 contract to P_0 in $R \bowtie I$. Then by assumption $Q_0 = Q_1$ so we see that $Q = Q'$. Thus $R \subset T$ satisfies INC.

Conversely, suppose that $R \subset T$ satisfies INC. Let $Q_0 \subset Q_1$ be primes of $T \bowtie J$ both contracting to some prime P_0 in $R \bowtie I$. Note by Lemma 2.1.4 if Q_0 is of the first form (resp. the second form) of Lemma 1.5.8, then we can write P_0 in the first form (resp. the second form) by Lemma 2.1.4.

Suppose that $Q_0 = \{(q, q+j) | q \in Q, j \in J\}$ for some $Q \in Spec(T)$, so that we can write $P_0 = \{(p, p+i) | p \in P, i \in I\}$ where $P = Q \cap R$. If $Q_1 = \{(q', q'+j) | q' \in Q', j \in J\}$, then it follows that $Q' \cap R = P$ as well. Clearly $Q \subseteq Q'$ by construction, so since $R \subset T$ satisfies INC, we conclude that $Q = Q'$; thus $Q_0 = Q_1$. The reasoning when all three primes are of the second form follows the same logic, by symmetry.

Next suppose That Q_0 is of form 1, but Q_1 is of form 2. Say $Q_0 = \{(q, q+j) | q \in Q, j \in J\}$, $Q_1 = \{(q'+j, q') | q' \in Q', j \in J\}$ for some $Q, Q' \in Spec(T)$. Then by Lemma 2.1.4, we can write P_0 as $\{(p, p+i) | p \in P, i \in I\}$ where $P = Q \cap R$, and as $\{(p'+i, p') | p', i \in I\}$ with $P' = Q' \cap R$. Comparing these two sets we see that $P \subseteq P'$ (by setting the i to zero in the second coordinates) and $P' \subseteq P$ (by setting the i to zero in the first coordinates). Thus $P = P' = Q \cap R = Q' \cap R$. By setting the j to zero in the second coordinates of their definitions of Q_0, Q_1 we see that $Q \subseteq Q'$. Since these contract to the same prime P in R, we must have $Q = Q'$, as $R \subset T$ satisfies INC. Thus by our definitions of Q_0, Q_1, we have the equality

$$\{(q, q+j) | q \in Q, j \in J\} = \left\{(q', q'+j) | q' \in Q', j \in J\right\} \subseteq$$

$$\left\{(q'+j, q') | q' \in Q', j\right\}.$$

Looking at the second coordinates of the last two sets we see that $Q' + J \subseteq Q'$, so that in particular, $J \subseteq Q'$. Thus by Lemma 1.5.8, all three of the above sets are equal; that is, $Q_0 = Q_1$.

Finally, suppose that Q_0 is of form 2, but Q_1 is of form 1. Say $Q_0 = \{(q+j, q) | q \in Q, j \in J\}$, $Q_1 = \{(q', q'+j) | q' \in Q', j \in J\}$ for some $Q, Q' \in Spec(T)$. Then by Lemma 2.1.4, we can write P_0 as $\{(p+i, p) | p \in P, i \in I\}$ where $P = Q \cap R$, and as $\{(p', p'+i) | p', i \in I\}$ with $P' = Q' \cap R$. Comparing these two sets we see that $P \subseteq P'$ (by setting the i to zero in the first coordinates) and $P' \subseteq P$ (by setting the i to zero in the second coordinates). Thus $P = P' = Q \cap R = Q' \cap R$. By setting the j to zero in the first coordinates of their definitions of Q_0, Q_1 we see that $Q \subseteq Q'$. Since these contract to the same prime P in R, we must have $Q = Q'$, since $R \subset T$ satisfies INC. Thus by our definitions of Q_0, Q_1, we have the equality

$$\{(q+j,q)|q \in Q, j \in J\} = \left\{(q'+j,q')|q' \in Q', j \in J\right\} \subseteq$$

$$\left\{(q',q'+j)|q' \in Q', j\right\}.$$

Looking at the first coordinates of the last two sets we see that $Q' + J \subseteq Q'$, so that in particular, $J \subseteq Q'$. Thus by Lemma 1.5.8, all three of the above sets are equal; that is, $Q_0 = Q_1$.

As we have covered all possible cases, it follows that $R \bowtie I \subset T \bowtie J$ satisfies INC.

$\square$

Proposition 6.3.6. *Let $R \subseteq T$ be rings with ideals I, J, respectively, such that $I \subseteq J$. Then the extension $R \bowtie I \subseteq T \bowtie J$ satisfies going-up if and only if the extension $R \subseteq T$ satisfies going-up.*

Proof. First we will assume that $R \subseteq T$ satisfies going-up. Suppose $P_0 \subseteq P_1 \in Spec(R \bowtie I)$ and that $Q_0 \in Spec(T \bowtie J)$ contracts to P_0 in $R \bowtie I$. We wish to find a $Q_1 \in Spec(T \bowtie J)$ contracting to P_1 in $R \bowtie I$ such that $Q_0 \subseteq Q_1$.

First we will assume that Q_0 is a prime of the first form described in Lemma 1.5.8. Note by Lemma 2.1.4 that we can also assume P_0 is written in this form. Now, if P_1 is also in the form then we have

$$Q_0 = \{(q, q+j)|q \in Q, j \in J\}$$

$$P_0 = \{(p, p+i)|p \in P, i \in I\}$$

$$P_1 = \left\{(p', p'+i)|p' \in P', i \in I\right\}$$

where $Q \in Spec(T)$, $P, P' \in Spec(R)$, and $Q \cap R = P$. Since $P_0 \subseteq P_1$ it is clear that $P \subseteq P'$. Then, as $R \subset T$ satisfies GU, we can find a prime Q' of T contracting to P' in

R, with $Q \subseteq Q'$. Then note that $Q_1 := \{(q', q'+j) | q' \in Q', j \in J\}$ is a prime in $T \bowtie J$ by Lemma 1.5.8. Further, $Q_0 \subseteq Q_1$ and $Q_1 \cap R \bowtie I = P_1$ (by Lemma 2.1.4). Thus Q_1 gives our desired prime.

If we assume that Q_0, P_0, P_1 are all of the second prime form (in the sense of Lemma 1.5.8), then a similar argument gives a prime $Q_1 = \{(q'+j, q') | q' \in Q', j \in J\}$ to complete the diagram.

Next assume that Q_0 has form 1 (thus we can assume P_0 does as well by Lemma 2.1.4), and that P_1 has form 2. Say $Q_0 = \{(q, q+j) | q \in Q, j \in J\}$, $P_0 = \{(p, p+i) | p \in P, i \in I\}$, and $P_1 = \{(p'+i, p') | p' \in P', i \in I\}$ for some primes $Q \in Spec(T), P' \in Spec(R)$ and with $P = Q \cap R$. Then by assumption we have the inclusion $P_0 \subseteq P_1$; that is,

$$\{(p, p+i) | p \in P, i \in I\} \subseteq \left\{(p'+i, p') | p' \in P', i \in I\right\}.$$

Then setting the p to zero in the second coordinates we see that $I \subseteq P'$. But then by Lemma 1.5.8 we can write P_1 in form 1, so that we reduce to the original case where all three primes involved were of form 1.

Finally, assume that Q_0 has form 2 (and so we can write P_0 in form 2 as well by Lemma 2.1.4), and that P_1 has form 1. Let us explicitly write these sets, Say $Q_0 = \{(q+j, q) | q \in Q, j \in J\}$, $P_0 = \{(p+i, p) | p \in P, i \in I\}$, and $P_1 = \{(p', p'+i) | p' \in P', i \in I\}$ for some primes $Q \in Spec(T), P' \in Spec(R)$ and with $P = Q \cap R$. Again by assumption we have the inclusion $P_0 \subseteq P_1$; in this case, the inclusion looks like

$$\{(p+i, p) | p \in P, i \in I\} \subseteq \left\{(p', p'+i) | p' \in P', i \in I\right\}.$$

Now setting the p to zero in the first coordinate, we can see that $I \subseteq P'$. Again by Lemma 1.5.8 we are reduced to an earlier case, this time the case where all three primes involved are of form 2. Since all cases have now been covered, we conclude that $R \bowtie I \subseteq T \bowtie J$ satisfies going-up.

For the converse, suppose that $R \bowtie I \subseteq T \bowtie J$ satisfies going-up, and let $Q \in$

$Spec(T), P, P' \in Spec(R)$, with $Q \cap R = P$. We wish to find a prime in T containing Q_0 and contracting to P_1 in R. By Lemma 1.5.8, the set $Q_0 := \{(q, q+j) | q \in Q, j \in J\}$ is a prime ideal of $T \bowtie J$ and the sets $P_0 := \{(p, p+i) | p \in P, i \in I\}$, $P_1 := \{(p', p'+i) | p' \in P', i \in I\}$ are prime ideals of $R \bowtie I$. Note that $Q_0 \cap (R \bowtie I) = P_0$ and $P_0 \subseteq P_1$. Thus by assumption we can find a prime ideal Q_1 of $T \bowtie J$ with $Q_0 \subseteq Q_1$ and $Q_1 \cap (R \bowtie I) = P_1$. If Q_1 is a prime of the first form of Lemma 1.5.8, say $Q_1 = \{(q', q'+j) | q' \in Q', j \in J\}$ for some $Q' \in Spec(T)$, then it is easy to see by Lemma 2.1.4 that Q' will fill the requirements for our sought prime. Finally, assume that Q_1 is of the second form of Lemma 2.1.4, say $Q_1 = \{(q'+j, q') | q' \in Q', j \in J\}$ for some $Q' \in Spec(T)$. Then $Q_1 \cap (R \bowtie I) = P_1$, so

$$\left\{(q'+i, q') | q' \in Q' \cap R, i \in I\right\} = \left\{(p', p'+i) | p' \in P', i \in I\right\}.$$

By setting the i all to zero in the first coordinate we see that $Q' \cap R \subseteq P'$, and by setting the i to zero in the second coordinate we see that $P' \subseteq Q' \cap R$. Thus $Q' \cap R = P$. Similarly, by setting the j to zero in the second coordinate in the definition of Q_0 (and noting that $Q_0 \subseteq Q_1$) we see that $Q \subseteq Q'$. It follows that we can take Q' as our desired prime. Thus $R \subseteq T$ satisfies going-up.

□

We will conclude with an investigation into when an extension of bowtie rings $R \bowtie I \subseteq T \bowtie J$ satisfies going-down. These conditions are considerably more complicated than in the cases for LO, INC, and GU. This as is often the case; for instance, every integral extension will satisfy LO, INC, and GU, but none of this information is enough to imply GD.

We will also note that every flat extension of rings satisfies going-down (cf. [M2, p. 33]); thus we need conditions on I and J that are satisfied by (though not necessarily equivalent to) the condition $J = IT$ in Theorem 4.2.12. We find the appropriate conditions in the following theorem.

Theorem 6.3.7. *Let $R \subseteq T$ be an extension of rings with ideals $I \subseteq J$, respectively. The ring extension $R \bowtie I \subseteq T \bowtie J$ satisfies going-down if and only if $R \subseteq T$ satisfies going-down and for every pair of primes $Q \subseteq Q'$ of T with $I \nsubseteq Q$ and $J \nsubseteq Q'$ it follows that $I \nsubseteq Q'$.*

Proof. Suppose that $R \bowtie I \subseteq T \bowtie J$ satisfies going-down; we will show that $R \subseteq T$ does as well. Let $P \subseteq P'$ be primes of R and Q' a prime of T such that $Q' \cap R = P'$. We must find a prime Q of T, contained in Q' and contracting to P in R. By Lemma 1.5.8 we can construct primes $P_0 = \{(p, p+i) | p \in P, i \in I\} \subseteq P_1 = \{(p', p'+i) | p' \in P', i \in I\}$ of $R \bowtie I$, and $Q_1 = \{(q, q+j) | q \in Q, j \in J\}$ of $T \bowtie J$. By assumption we can now find a prime $Q_0 \subseteq Q_1$ of $T \bowtie J$ such that $Q_0 \cap (R \bowtie I) = P_0$.

By Lemma 1.5.8, Q_0 may come in one of two forms. If the first form, $Q_0 = \{(q, q+j) | q \in Q, j \in J\}$, then $Q \subseteq Q'$ clearly, and $Q_0 \cap (R \bowtie I) = \{(q, q+i) | q \in Q \cap R, i \in I\}$ so $Q \cap R = P$, and we can take Q as our desired prime. If Q_0 is of the second form, $Q_0 = \{(q+j, q) | q \in Q, j \in J\}$ then the assumption that $Q_0 \cap (R \bowtie I) = P_0$ gives that $\{(q+i, q) | q \in Q \cap R, i \in I\} = \{(p, p+i) | p \in P, i \in I\}$. Setting all the i to zero in the left side of this equation gives that $Q \cap R \subseteq P$ and doing the same in the right side gives $P \subseteq Q \cap R$, so $Q \cap R = P$. Similarly, setting all the j to zero in the definition of Q_0 (and recalling that $Q_0 \subseteq Q_1$) we see that $Q \subseteq Q'$. Thus we can take this Q as the prime we seek, and it follows that $R \subseteq T$ satisfies going-down.

We may assume that $R \subseteq T$ satisfies going-down for the remainder of the proof. Under this assumption, we only have to prove that $R \bowtie I \subseteq T \bowtie J$ satisfies going-down if and only if the conditions for I and J (in the statement of the theorem) are met.

Thus, suppose that for any primes $Q \subseteq Q'$ of T, $I \nsubseteq Q$ and $J \nsubseteq Q'$ imply that $I \nsubseteq Q'$. Let $P_0 \subseteq P_1$ be primes of $R \bowtie I$ and let $Q_1 \in \mathit{Spec}(T \bowtie J)$ contract to P_1 in $R \bowtie I$. First assume that all three of these primes have form (1) in the sense of Lemma 1.5.8. Say $P_0 = \{(p, p+i) | p \in P, i \in I\}$, $P_1 = \{(p', p'+i) | p' \in P', i \in I\}$, and $Q_1 = \{(q', q'+j) | q' \in Q', j \in J\}$. Then we must have $P \subseteq P'$, and by Lemma 2.1.4 that $Q' \cap R =$

P'. By assumption then we can find a prime Q of T contracting to P in R such that $Q \subseteq Q'$. Now take the prime $Q_0 = \{(q, q + j) | q \in Q, j \in J\}$ of $T \bowtie J$ to complete the going-down diagram. The case where all three of the given primes have form (2) (in Lemma 1.5.8) is similar.

By Lemma 2.1.4 we may assume that P_1 and Q_1 have the same form. Thus we only have two remaining situations to consider: where P_1 and Q_1 have form (1) but P_0 has form (2), and where P_1 and Q_1 have form (2) but P_0 has form (1). These situations are symmetric, so we will only consider the first, say $P_0 = \{(p + i, p) | p \in P, i \in I\} \subseteq P_1 = \{(p', p' + i) | p' \in P', i \in I\}$, and $Q_1 = \{(q', q' + j) | q' \in Q', j \in J\}$. Since $P_0 \subseteq P_1$ we see that $P \subseteq P'$. Further, by Lemma 2.1.4 it is clear that $Q' \cap R = P'$. Since $R \subseteq T$ satisfies going-down, we can find a prime Q of T contracting to P in R with $Q \subseteq Q'$. We now have two possible cases to consider.

Case 1: $I \subseteq Q$.

Set $Q_0 = \{(q, q + j) | q \in Q, j \in J\}$, and note that $Q_0 \subseteq Q_1$. Further, since $Q_0 \cap (R \bowtie I) = \{(p, p + i) | p \in P, i \in I\} = \{(p + i, p) | p \in P, i \in I\} = P_0$ (by Lemma 1.5.8) we have found our desired prime Q_0.

Case 2: $I \nsubseteq Q$.

Note that $P_1 = Q_1 \cap (R \bowtie I) = \{(q', q' + i) | q' \in Q' \cap R, i \in I\}$. The inclusion $P_0 \subseteq P_1$ thus implies that $P \subseteq Q' \cap R \subseteq Q'$. Our assumption (on the converse of this proof) now gives that $J \subseteq Q'$ or $I \nsubseteq Q'$.

First suppose that $J \subseteq Q'$. Since $P \subseteq P' = Q' \cap R$ and $R \subseteq T$ satisfies going-down, we can find a prime Q of T contracting to P in R such that $Q \subseteq Q'$. Set $Q_0 = \{(q + j, q) | q \in Q, j \in J\}$. Then Q_0 contracts to P_0 in $R \bowtie I$. Pick any $(q + j, q) \in Q_0$, and note that $(q + j, q) = (q + j, (q + j) - j) \in Q_1$ since $q + j \in Q'$. Thus $Q_0 \subseteq Q_1$, so Q_0 gives the needed prime.

Now instead suppose that $I \nsubseteq Q'$. We will find a contradiction, showing that this subcase is impossible. Currently, $P_1 = \{(p', p' + i) | p' \in P', i \in I\}$. Note that P_0 contains all elements of the form $(i,0)$ where $i \in I$. Since $P_0 \subseteq P_1$ it follows that $I \subseteq P' = Q' \cap R \subseteq Q'$.

For the converse, suppose that there exist primes $Q \subseteq Q'$ of T such that $I \nsubseteq Q$ and $J \nsubseteq Q'$ but $I \subseteq Q'$. We will show that going-down does not hold for the extension $R \bowtie I \subseteq T \bowtie J$. Consider the primes $P_0 = \{(p, p+i) | p \in P\}$ and $Q_1 = \{(q'+j, q') | q' \in Q', j \in J\}$ of $R \bowtie I$ and $T \bowtie J$, respectively, where $P := Q \cap R$. Then $Q_1 \cap (R \bowtie I) = \{(p'+i, p') | p' \in P' = Q' \cap R, i \in I\} = \{(p', p'+i) | p' \in P', i \in I\}$ by Lemma 2.1.4 and [D, Proposition 5] (since $I \subseteq Q'$). Since $P \subseteq P'$ (that is, $Q \cap R \subseteq Q' \cap R$) we now have that $P_0 \subseteq P_1$. If $R \bowtie I \subseteq T \bowtie J$ satisfies going-down, then we should now be able to find a prime Q_0 of $T \bowtie J$ contained in Q_1 and contracting to P_0 in $R \bowtie I$. We will show that this is impossible.

Suppose we are able to find such a Q_0. Then $P_0 \subseteq Q_0$ necessarily. Now for some prime Q of T, Q_0 has the form $\{(q, q+j) | q \in Q, j \in J\}$ or $\{(q+j, q) | q \in Q, j \in J\}$. Assume it has the first form. Since $J \nsubseteq Q'$ we can find a $j \in J \backslash Q'$. But then $(0, j) \in Q_0 \backslash Q_1$, contradicting that $Q_0 \subseteq Q_1$. Now assume it has the second form. Since Q_0 contracts to P_0 in $R \bowtie I$ we have by Lemma 2.1.4 that $\{(q+i, q) | q \in Q \cap R, i \in I\} = \{(p, p+i) | p \in P, i \in I\}$. Note that the left set contains all elements in $R \bowtie I$ of the form $(i, 0)$, which implies that $I \subseteq P = Q \cap R$, contradicting the assumption that $I \nsubseteq Q$. It follows that going-down does not hold.

□

Corollary 6.3.8. *Let $R \subseteq T$ be rings with ideals $I \subseteq J$, respectively, such that $R \subseteq T$ satisfies going-down. If $I \subseteq Nil(R)$, then the extension $R \bowtie I \subseteq T \bowtie J$ satisfies going-down.*

Proof. Since any nilpotent element of R is a nilpotent element of T, then I is contained in $Nil(T)$. Thus I is contained in every prime Q of T by Lemma 6.3.3, whence we have by Theorem 6.3.7 that $R \bowtie I \subset T \bowtie J$ satisfies going down.

□

Since $R \cong R^{\Delta} := R \bowtie 0$ the following is now immediate.

Corollary 6.3.9. *Let I be an ideal of a ring R. Then the extension $R \subset R \bowtie I$ satisfies going-down.*

We now have an easy way to construct such an extension of bowtie rings $R \bowtie I \subset R \bowtie J$ where going-down holds: simply choose an ideal I contained in $Nil(R)$. To avoid trivialities and show that the necessary and sufficient condition given in the theorem does not always hold, we give the following simple example.

Example 6.3.10. In Theorem 6.3.7, set $R = T = \mathbb{Z}[X]$, $Q = (0)$, $I = Q' = (X)$, and $J = (2, X)$. Then (by the theorem, with Corollary 6.1.3) the ring extension $\mathbb{Z}[X] \bowtie (X) \subset \mathbb{Z}[X] \bowtie (2, X)$ is an integral extension that does not satisfy going-down.

It is not hard to see that we could construct a similar example with any ring R where $dim(R) \geq 2$. We express this with the following corollary.

Corollary 6.3.11. *Let $I \subset J$ be ideals of a ring R. If $I \in Spec(R) \setminus Min(R)$, then the extension $R \bowtie I \subset R \bowtie J$ is an integral extension that does not satisfy going-down.*

Proof. In the statement of Theorem 6.3.7, set Q' equal to I and set Q equal to any prime that I properly contains. The integrality follows from Corollary 6.1.3.

□

Corollary 6.3.12. *Let $I \subset J$ be ideals of a ring R. If I is not prime and there are no primes lying between I and J (via set inclusion), then $R \bowtie I \subseteq R \bowtie J$ satisfies going-down.*

Finally, we can use simple bowtie rings to give examples of integral extensions that are minimal and satisfy GD (as well as LO, INC, and GU by [K, Theorem 44]).

Corollary 6.3.13. *Let $I \subset J$ be ideals of a chained ring R. If I is not prime and J/I is simple (as an $R-$module) then $R \bowtie I \subset R \bowtie J$ is an integral minimal extension that satisfies going-down.*

Proof. Integrality follows from Corollary 6.1.3, minimality from Corollary 3.2.5. Regarding going-down, note that for any prime Q of R, the inclusion $I \subseteq Q$ implies that $J \subseteq Q$.

□

Even if we do not assume minimality, we can still construct an integral $\lambda-$extension that satisfies going-down, citing Corollary 3.2.8.

Example 6.3.14. The ring extension $\mathbb{Z}_{2\mathbb{Z}} \bowtie 4\mathbb{Z}_{2\mathbb{Z}} \subset \mathbb{Z}_{2\mathbb{Z}} \bowtie 2\mathbb{Z}_{2\mathbb{Z}}$ is a minimal extension, it is integral (so satisfies LO, INC, and GD), and it satisfies going-down. If we replace $4\mathbb{Z}_{2\mathbb{Z}}$ with any other ideal of $\mathbb{Z}_{2\mathbb{Z}}$ properly contained in $4\mathbb{Z}_{2\mathbb{Z}}$, then this extension it is a non-minimal $\lambda-$extension that is integral and satisfies going-down.

References

[BDM] P. Bhattacharjee, K. M. Drees, and W. W. McGovern, *Extensions of commutative rings*, Topology and its Applications **158** (2011), 1802-1814.

[CDL] P.-J. Cahen, D.E. Dobbs, and T.G. Lucas, *Characterizing Minimal Ring Extensions*, Rocky Mountain Math. J. **41 (4)** (2011), 1081-1125.

[CM] M. Chhiti and N. Mahdou, *Some Homological Properties of Amalgamated Duplication of a Ring Along and Ideal*, Bulletin of the Iranian Mathematical Society **38 (2)** (2012), 507-515.

[C] A.L.S. Corner , *On endomorphism rings of primary abelian groups*, Quart. J. Math. Oxford **20** (1969), 277-296.

[D] M. D'Anna, *A construction of Gorenstein rings*, J. Algebra **306** (2006), 507-519.

[DFF] M. D'Anna, C.A. Finocchiaro, and M. Fontana, *Properties of chains of prime ideals in an amalgamated algebra along an ideal*, J. Pure Appl. Algebra **214** (2010), 1633-1641.

[DFF2] M. D'Anna, C.A. Finocchiaro, and M. Fontana, *Amalgamated algebras along an ideal*, in "Commutative Algebra and Applications", Proceedings of the Fifth International Fez Conference on Commutative Algebra and Applications, Fez, Morocco, 2008, W. de Gruyter Publisher, Berlin, 2009.

[DF] M. D'Anna, M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **6** (2007), 443-459.

[DF2] M. D'Anna, M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat. **45** (2) (2007), 241-252.

[D2] D. Dobbs, *Every Commutative Ring Has a Minimal Extension*, Communications in Algebra **34** (2006), 3875-3881.

[DS] D. Dobbs and J. Shapiro, *A classification of the minimal ring extensions of an integral domai*n, J. Algebra **305 (1)** (2006), 185-193.

[DS2] D. Dobbs and J. Shapiro, *A classification of the minimal ring extensions of certain commutative rings*, J. Algebra **308 (2)** (2007), 800-821.

[DS3] D. Dobbs and J. Shapiro, *Normal pairs with zero-divisors*, J. Algebra Appl. **10** (2011), 335-356.

[DS4] D. Dobbs and J. Shapiro, *On bowtie rings, universal survival rings, and universal lying-over rings*, Rendiconti del Circolo Matematico di Palermo **61 (1)** (2012), 123-131.

[DS5] D. Dobbs and J. Shapiro, *Universal lying-over rings*, Comm. Algebra **36** (2008), 2895-2904.

[DS6] D. Dobbs and J. Shapiro, *Universal quasi-lying over rings*, Houston journal of mathematics, **37 (2)** (2011), 351-359.

[D3] J. L. Dorroh, *Concerning adjunctions to algebras*, Bull. Amer. Math. Soc. **38** (1932), 85-88.

[DM] T.J. Dorsey and Z. Mesyan, *On minimal extensions of rings*, Comm. Algebra 37 (2009), 3463-3486.

[F] C.A. Finocchiaro, Amalgamation of algebras and the ultrafilter topology on the Zariski space of valuation overrings of an integral domain (Ph.D. dissertation) (2011).

[FO] D. Ferrand and J.P. Olivier, *Homomorphismes Minimaux d'Anneaux*, J. Algebra **16** (1970), 461-471.

[G] M.S. Gilbert, *Extensions of Commutative Rings With Linearly Ordered Intermediate Rings* (Ph.D. dissertation) (1996).

[G2] R. Gilmer, Multiplicative Ideal Theory, Dekker, New York, 1972.

[GH] R. Gilmer, J. A. Huckaba, $\Delta-$*Rings*, J. Algebra **28** (1974), 414-432.

[G3] S. Glaz, Commutative Coherent Rings, Springer-Verlag, Lecture Notes in Mathematics **1371**, 1989.

[H] J. Huckaba, Commutative rings with zero divisors, M. Dekker, New York, 1988.

[HS] C. Huneke and I. Swanson, Integral Closure of Ideals, Rings, and Modules, Cambridge University Press, Cambridge, 2006.

[K] I. Kaplansky, Commutative Rings, Rev. Ed., Univ. Chicago Press, Chicago, 1974.

[L] D. Lazard, *Autour de la platitude*, Bull. Soc. Math. France **97** (1969), 81-128.

[M] H. Matsumura, Commutative Ring Theory, Cambridge University Press, Cambridge, 1987.

[M2] H. Matsumura, Commutative Algebra, Benjamin, New York, 1970.

[N] M. Nagata, *The theory of multiplicity in general local rings*, Proc. Intern. Symp. Tokyo-Nikko 1955, Sci. Council of Japan, Tokyo, 1956, 191-226.

[N2] M. Nagata, Local Rings, Interscience, New York, 1962.

[PP] G. Picavet and M. Picavet-L'Hermitte, *About minimal morphisms*, in Multiplicative Ideal Theory in Commutative Algebra, Springer, New York, 2006.

[R] J. Rotman, An Introduction to Homological Algebra, 2nd ed., Press, New York, 1979.

[S1] J. Shapiro, *On a construction of Gorenstein rings proposed by M. D'Anna*, J. Algebra **323** (2010), 1155-1158.

[S2] T.S. Shores, *Loewy series of modules*, J. reine angew. Math. **265** (1974), 183-200.

Biography

Tim S. Long received his A.S. degree in Science from Northern Virginia Community College in 2002. He recieved his B.S. and M.S. degrees in Mathematics from George Mason University in 2005 and 2012, respectively. He has worked as a Graduate Teaching Assistant with George Mason University since 2006.